LA

Linear Algebra

LA.1 EQUIVALENCE AND INVARIANTS

Definition LA.1.1 – Denote with X a set and with E an equivalence relation defined on X . Denote then with S a second set and with $f : X \to S$ a function. If x' and x'' are two elements of X and f is such that $x' \, E \, x''$ implies $f(x') = f(x'')$, f is called an *invariant* for E. Moreover if $f(x') = f(x'')$ implies $x' \, E \, x''$, f is called a *complete invariant* for E.

If f is a complete invariant for E, then all elements of X belonging to the same equivalence class have the same image in f; moreover, these classes coincide exactly with the inverse images in f of the elements of the image (or range) of f. There exists, therefore, a bijection between the quotient set X/E and the image of a complete invariant for E.

Definition LA.1.2 – A set of invariants $f_1, \ldots, f_n$, $f_i : X \to S_i$ for E is called a *complete set of invariants* for E if the function $f = (f_1, \ldots, f_n) : X \to S_1 \times \ldots \times S_n$ defined by $x \to (f_1(x), \ldots, f_n(x))$ is a complete invariant for E.

Definition LA.1.3 – A set of invariants for E, $f_1, \ldots, f_n$, $f_i : X \to S_i$, will be called *independent* if the associated invariant $f = (f_1, \ldots, f_n) : X \to S_1 \times \ldots \times S_n$ is surjective.

This condition implies that no invariant f_i can be expressed as a function of the others. This last condition, however, is weaker than the definition of independence that has been given. A complete set of independent invariants for E is also called a *basis* for E on X.

Lemma LA.1.1 – Let $f : X \to S$ be a complete surjective invariant for E. Then every other invariant for E can be uniquely computed from f.

Proof: Let $f : X \to S$ and $g : X \to T$ be, respectively, a complete surjective invariant and a generic invariant for E. Commutativity in the diagram of Fig. LA.1.1

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & T \\ & S \arrow[ur, "h"'] & \end{tikzcd}$$

Figure LA.1.1

can be obtained if and only if for every element s of S the function h is defined as $h(s) = g(x)$ where x is any element of X such that $f(x) = s$. Since f is complete and surjective and g is an invariant, h is well defined for all elements of S.

Corollary LA.1.1 – Let $f : X \to S$ be a complete set of independent invariants for E. Then every other invariant for E can be uniquely computed from f.

Property LA.1.1 – Let $f : X \to S$ be a complete set of independent invariants for E. If $g : S \to R$ is a bijection, then $h = g \cdot f : X \to R$ is a complete set of independent invariants for E.

CANONICAL FORMS FOR EQUIVALENCE RELATIONS

Definition LA.1.4 – Let E be an equivalence relation on X. A subset C of X is called a set of *canonical forms* for E if every $x \in X$ is equivalent under E to one and only one element of C; this element is *the* canonical form of x. The function $g : X \to C$ thus defined is therefore a complete invariant for E. Obviously g can be assumed surjective without loss of generality.

COMPLETE SETS OF INDEPENDENT INVARIANTS AND CANONICAL FORMS

Let $f : X \to S$ be a complete set of independent invariants and C a set of canonical forms for E. Then (Corollary LA.1.1) there exists a unique function $h : S \to C$ such that $g = h \cdot f$. Since g is complete, h is a bijection. Moreover if $i : C \to X$ is the injection $i(c) = c$, it follows that $h^{-1} = f \cdot i$. The following theorem has thus been proved.

Theorem LA.1.1 – Let C be a set of canonical forms for an equivalence relation E on X and f a complete set of independent invariants for E. Then there exists a unique bijection between C and the image of f.