ID3

ARX Identification

3.17 MULTIVARIABLE ARX MODELS AND PREDICTORS

Multivariable ARX models can be introduced starting from the decomposition shown in Figure 3.1.2 and substituting the scalar transfer function $p(z)/q(z)$ with any decomposition $Q(z)^{-1}P(z)$ of the transfer matrix $G(z)$ with $Q(z)$ and $P(z)$ left coprime. We obtain the scheme reported in Figure 3.17.1 where $e(t) \in \mathcal{R}^m$ is independent of $u(t)$ and has components given by independent white noises with variances $\sigma_{e1}^2, \dots, \sigma_{em}^2$, i.e.

$$\operatorname{cov} e(t) = \mathrm{E}\left[\, e(t)\, e(t)^T \,\right] = \operatorname{diag}\left[\, \sigma_{e1}^2, \dots, \sigma_{em}^2 \,\right]. \tag{3.17.1}$$

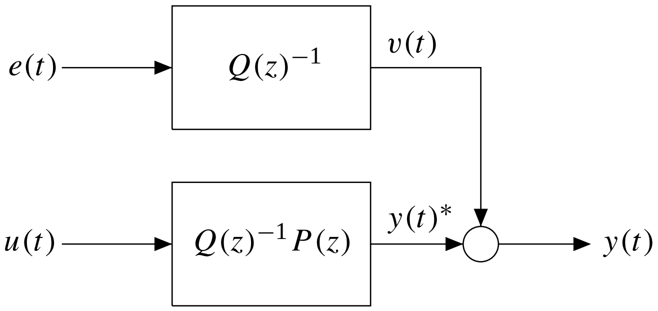

Figure 3.17.1 – Partition of a multivariable ARX process

Note that the term z^n present in Figure 3.1.2 has been omitted in Figure 3.17.1 in that irrelevant (independent white noise sequences can be freely translated along the time axis). Differently from the scalar case, it is possible to consider several decompositions for $G(z)$. The only decompositions of interest in our context are the identifiable ones, like those characterized by minimal parameterizations. Considering canonical pairs (ST.4.9) we obtain models constituted by the m relations

$$\begin{aligned} y_i(t+\nu_i) &= \sum_{j=1}^{m}\sum_{k=1}^{\nu_{ij}} \alpha_{ijk}\, y_j(t+k-1) \\ &\quad + \sum_{j=1}^{r}\sum_{k=1}^{\nu_i} \beta_{ijk}\, u_j(t+k-1) + e_i(t+\nu_i) \quad (i = 1, \dots, m) \end{aligned} \tag{3.17.2}$$

where the components of $e(t)$ have been considered at times $t + \nu_i$ for the already mentioned irrelevance of their translation along the time axis. Model (3.17.2) can also be written in the polynomial form

$$Q(z)\,y(t) = P(z)\,u(t) + D(z)\,e(t) \tag{3.17.3}$$

where the entries of $Q(z)$ and $P(z)$, defined by (ST.4.10)–(ST.4.12), satisfy relations (ST.4.18) and $D(z)$ is given by

$$D(z) = \operatorname{diag}\left[\, z^{\nu_1},\ \ldots,\ z^{\nu_m} \right]. \tag{3.17.4}$$

Models (3.17.2)–(3.17.3) are identifiable and have all advantages of minimal parameterizations but have also some drawbacks due essentially to the fact that they are constituted by m non synchronous (unless all indices ν_i are equal) forward relations that prevent a direct transformation to backward notations and a direct construction of predictors. For these reasons they can not be properly considered as ARX representations. To obtain minimally parameterized ARX models define the non singular and non unimodular matrix

$$M(z) = \operatorname{diag}\left[\, z^{\Delta\nu_1},\ \ldots,\ z^{\Delta\nu_m} \right] \tag{3.17.5}$$

where $\Delta\nu_i$ denotes the difference $\nu_M - \nu_i$ ($\nu_M = \max_i(\nu_i)$) and consider the equivalent polynomial model

$$Q(z)^* y(t) = P(z)^* u(t) + D(z)^* e(t) \tag{3.17.6}$$

where

$$Q(z)^* = M(z)\,Q(z) = Q^*_{\nu_M+1}\,z^{\nu_M} - Q^*_{\nu_M}\,z^{\nu_M-1} - \ \ldots\ - Q^*_1 \tag{3.17.7a}$$

$$P(z)^* = M(z)\,P(z) = P^*_{\nu_M}\,z^{\nu_M-1} + \ \ldots\ + P^*_2 z + P^*_1 \tag{3.17.7b}$$

$$D(z)^* = M(z)\,D(z) = z^{\nu_M}\,I\,. \tag{3.17.7c}$$

Models (3.17.3) and (3.17.6) are different decompositions of the same transfer matrix but are not strictly equivalent since $\deg\det Q(z)^* > \deg\det Q(z)$ because (3.17.6) includes the additional non reachable dynamics defined by $M(z)$. Model (3.17.6) can be written in the expanded ARX form

$$y(t+\nu_M) = Q^{*-1}_{\nu_M+1}\left[\,\sum_{i=1}^{\nu_M} Q^*_i\,y(t+i-1) + \sum_{i=1}^{\nu_M} P^*_i\,u(t+i-1)\,\right] + e^*(t+\nu_M) \tag{3.17.8}$$

where

$$e^*(t) = Q^{*-1}_{\nu_M+1}\,e(t) \tag{3.17.9}$$

and can be rewritten in backward ARX notation as follows

$$y(t) = Q^{*-1}_{\nu_M+1} \left[\sum_{i=1}^{\nu_M} Q^*_{\nu_M+1-i}\, y(t-i) + \sum_{i=1}^{\nu_M} P^*_{\nu_M+1-i}\, u(t-i) \right] + e^*(t). \tag{3.17.10}$$

ARX models (3.17.8) and (3.17.10) exhibit equation errors given by $e^*(t)$; the corresponding optimal predictor is

$$y(t|t-1) = Q^{*-1}_{\nu_M+1} \left[\sum_{i=1}^{\nu_M} Q^*_{\nu_M+1-i}\, y(t-i) + \sum_{i=1}^{\nu_M} P^*_{\nu_M+1-i}\, u(t-i) \right]. \tag{3.17.11}$$

It is important to note that models (3.17.6) and (3.17.2) share the same minimal parameterization that defines univocally the parameterizations of ARX models (3.17.8), (3.17.10) and of predictor (3.17.11).

Remark 3.17.1 – $Q^*_{\nu_M+1}$, because of relations (ST.4.18), is a lower left triangular matrix with unitary elements on its main diagonal and is, consequently, always non singular and well conditioned ($\det Q^*_{\nu_M+1} = 1$).

Remark 3.17.2 – The vector $e(t)$ appearing in (3.17.2) and (3.17.3) cannot be directly considered as an equation error because models (3.17.2) and (3.17.3), while defining a minimal parameterization for ARX models (3.17.8) and (3.17.10), do not constitute, *per se*, ARX models. Because of (3.17.9), the link between the covariance matrices of $e(t)$ and $e^*(t)$ is

$$\operatorname{cov} e^*(t) = Q^{*-1}_{\nu_M+1} \operatorname{cov} e(t) \left(Q^{*-1}_{\nu_M+1}\right)^T. \tag{3.17.12}$$

Note that the components of $e^*(t)$, differently from those of $e(t)$, are correlated white processes.

Remark 3.17.3 – The canonical model (3.17.2) is, differently from (ST.4.17), purely dynamic as always happens with equation error models in view of their predictive applications.

Remark 3.17.4 – It can be questioned which rationale exists behind the definition of ARX multivariable models using, as intermediate tool, model (3.17.2) characterized by a vector of equation errors $e(t)$ with independent components and whether models of this kind are realistic or not. The second question is, in fact, immaterial because real processes never belong to the families of models used in their identification. The reply to first question is based on the observation that the unbiasedness of least squares estimates for multivariable ARX models and the minimization of cost function (3.18.14) require the independence of the components of $e(t)$ and that the peculiar stochastic environment of ARX models derives from least squares estimates and not *vice–versa*. The extension to the multivariable case of ARX models is thus nothing else than the extension to this case of least squares; the associated stochastic environment is only a consequence.

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