ID3

ARX Identification

3.5 BIAS AND CONSISTENCY OF LEAST SQUARES ESTIMATES

It has been shown that the least squares algorithm, given by expression (3.3.12), minimizes for ARX models, the mean square prediction error, when identifiability conditions are satisfied. It is now important to evaluate whether this algorithm gives an estimate, θ°, that converges, asymptotically, to the real parameters, θ^*, of the process that has generated the data. Consider the expression of the estimate bias, given by

$$b = \mathrm{E}\,[\theta^\circ] - \theta^*; \tag{3.5.1}$$

considering (3.3.12) and relation

$$y^\circ = H\theta^* + e^\circ \tag{3.5.2}$$

where

$$e^\circ = \big[\, e(n+1)\ \ldots\ e(L)\,\big]^T, \tag{3.5.3}$$

we obtain

$$\begin{aligned} b &= \mathrm{E}\big[\,(H^TH)^{-1}H^T(H\theta^* + e^\circ)\,\big] - \theta^* \\ &= \mathrm{E}\big[\,(H^TH)^{-1}H^TH - I\,\big]\theta^* + \mathrm{E}\big[\,(H^TH)^{-1}H^Te^\circ\,\big] \\ &= \mathrm{E}\big[\,(H^TH)^{-1}H^Te^\circ\,\big]. \end{aligned} \tag{3.5.4}$$

Expression (3.5.4) cannot be easily evaluated, for finite values of N, because both H and e° depend on the same stochastic process $e(\cdot)$; it is thus not possible to evaluate (3.5.4) as the product of the expected values of its factors. It is however possible to evaluate the limit to which the estimate θ° converges, in probability, for $N \to \infty$. Remembering that if A and B depend on the same stochastic process, $\mathrm{plim}\,[AB] = \mathrm{plim}\,[A]\,\mathrm{plim}\,[B]$, we obtain

$$\begin{aligned}
\text{plim}_{N\to\infty}\theta^\circ &= \text{plim}_{N\to\infty}\left[\,(H^T H)^{-1} H^T (H\theta^* + e^\circ)\,\right] \\
&= \theta^* + \text{plim}_{N\to\infty}\left[\,(H^T H)^{-1} H^T e^\circ\,\right] \\
&= \theta^* + \text{plim}_{N\to\infty}\left[\left(\frac{H^T H}{N}\right)^{-1}\frac{H^T e^\circ}{N}\right] \\
&= \theta^* + \text{plim}_{N\to\infty}\left[\left(\frac{H^T H}{N}\right)^{-1}\right]\text{plim}_{N\to\infty}\left[\frac{H^T e^\circ}{N}\right].
\end{aligned} \tag{3.5.5}$$

As discussed previously, when making reference to a SISO system, H^T can be partitioned as

$$H^T = \begin{bmatrix} H_n^T(y) \\ H_n^T(u) \end{bmatrix}; \tag{3.5.6}$$

the ith element of $H_n^T(y)e^\circ/N$ is given by

$$\left[\frac{H_n^T(y)\,e^\circ}{N}\right]_i = \frac{1}{N}\sum_{j=0}^{N-1} y(i+j)\,e(n+j+1) \qquad (i=1,\dots,n) \tag{3.5.7}$$

and, similarly, the ith element of $H_n^T(u)\,e^\circ/N$ is

$$\left[\frac{H_n^T(u)\,e^\circ}{N}\right]_i = \frac{1}{N}\sum_{j=0}^{N-1} u(i+j)\,e(n+j+1) \qquad (i=1,\dots,n)\,. \tag{3.5.8}$$

Since $e(\cdot)$ is a white process and index i in expressions (3.5.7) and (3.5.8) assumes values between 1 and n, the products on the right sides of (3.5.7) and (3.5.8) describe the sample covariance between the samples of $e(t)$ and previous input and output samples; then

$$\text{plim}_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} y(i+j)\,e(n+j+1) = 0 \quad (i=1,\dots,n) \tag{3.5.9}$$

$$\text{plim}_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} u(i+j)\,e(n+j+1) = 0 \quad (i=1,\dots,n) \tag{3.5.10}$$

$$\text{plim}_{N\to\infty}\left[\frac{H^T e^\circ}{N}\right] = 0. \tag{3.5.11}$$

It follows that

$$\text{plim}_{N\to\infty}\,\theta^\circ = \theta^* \,; \tag{3.5.12}$$

the least squares estimate of the parameters of an ARX model is thus asimptotically unbiased. Note however that, what has been proved does not establish that in any single identification experiment the bias will tend to zero when the lenght of the available sequences tends to infinity, but only that in an experiment of this kind, the limit in probability of the estimate coincides with the true parameters. The whiteness of $e(\cdot)$ allows, in fact, to give a stronger condition, i.e. to establish that

$$\lim_{N\to\infty}\left[\frac{H^T e^\circ}{N}\right] = 0 \tag{3.5.13}$$

with probability 1. An estimate with this property is defined *consistent*.

Remark 3.5.1 – It has been shown that the whiteness of $e(\cdot)$ determines the consistency of the least squares estimate. This condition is also necessary if

$$\text{plim}_{N\to\infty}\left[\left(\frac{H^T H}{N}\right)^{-1}\right] \neq 0; \tag{3.5.14}$$

we have seen that this happens when the input is persistently exciting of order n and n does not exceed the order of the system that has generated the data.

Remark 3.5.2 – It is important to observe that (3.5.13) implies, when $e(\cdot)$ is ergodic, relation

$$\text{E}\left[\, H^T e^\circ \,\right] = 0. \tag{3.5.15}$$

This condition will always be assumed as satisfied in the following.