LA

Linear Algebra

LA.3 EIGENVALUES AND EIGENVECTORS, S.V.D.

Given a real $n \times n$ matrix A, there is at least a complex scalar λ and an associated nonzero vector x such that

$$Ax = \lambda\, x \tag{LA.3.1}$$

or equivalently

$$\big(\lambda\, I - A\big)\, x = 0; \tag{LA.3.2}$$

λ is called an eigenvalue of A and x is the the associated eigenvector. Relation (LA.3.1) shows a peculiar property of eigenvectors: Ax has the same direction as x. The eigenvalues can be obtained by solving equation (LA.3.2) which admits nonzero solutions in x if and only if $(\lambda\, I - A)$ is singular, i.e. when

$$\det\big(\lambda\, I - A\big) = 0. \tag{LA.3.3}$$

Equation (LA.3.3) is called the characteristic equation of A; its n roots, $\lambda_1, \ldots, \lambda_n$, which are, in general, complex, are called the spectrum of A. The left side of equation (LA.3.3) is a n-degree polynomial in λ, $p(\lambda)$, called the characteristic polynomial of A.

Every eigenvalue λ_i $(i = 1, \ldots, n)$ corresponds to at least one nonzero eigenvector x_i. It can be noted that any nonzero multiple of an eigenvector is an eigenvector associated with the same eigenvalue; to avoid any ambiguity eigenvectors are often normalized, i.e. taken with unitary Euclidean norm. The eigenvalues of a matrix satisfy the following properties:

- If A is singular there exists at least one nonzero vector x such that $Ax = 0$; thus every singular matrix admits at least one null eigenvalue;
- If A is nonsingular, all its eigenvalues are nonzero and the eigenvalues of A^{-1} are their reciprocals;
- All eigenvalues of symmetric matrices are real. Symmetric matrices have n orthogonal eigenvectors that, when normalized, constitute an orthonormal basis of $\mathcal{R}^n$;

- $\mathrm{trace}(A) = \sum_{i=1}^{n} \lambda_i$;
- $\det(A) = \prod_{i=1}^{n} \lambda_i$.

The following theorem, known as Cayley-Hamilton theorem, describes a fundamental property of characteristic polynomials.

Theorem LA.3.1 Every square matrix annihilates its characteristic polynomial, i.e. $p(A) = 0$.

Definition LA.3.1 The minimal polynomial of A, $m(\lambda)$ is defined as the monic polynomial with minimal degree annihilated by A, i.e. such that $m(A) = 0$.

It is possible to show that the minimal polynomial of a matrix is unique and that it is a divisor of every polynomial $q(\lambda)$ annihilated by A. All zeros of $m(\lambda)$ are thus eigenvalues of A and it can be shown also that all distinct eigenvalues of A are zeros of $m(\lambda)$. Let us denote the characteristic polynomial of A as

$$p(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n; \tag{LA.3.4}$$

an efficient procedure to compute $p(\lambda)$ and, at the same time, the inverse of the polynomial matrix $(\lambda\, I - A)$

$$\left(\lambda\, I - A\right)^{-1} = \frac{\mathrm{adj}(\lambda\, I - A)}{\det(\lambda\, I - A)} = \frac{B_0\, \lambda^{n-1} + B_1\, \lambda^{n-2} + \ldots + B_{n-2}\, \lambda + B_{n-1}}{p(\lambda)} \tag{LA.3.5}$$

is given by the following Soriau–Leverrier algorithm.

Algorithm LA.3.1 – The coefficients α_i, $(i = 1, \ldots, n)$ of $p(\lambda)$ and the matrices B_i, $(i = 0, \ldots, n-1)$ in (LA.3.5) can be obtained by means of the following recursion:

$$\begin{aligned}
&B_0 = I && \alpha_1 = -\mathrm{tr} A \\
&B_1 = A\, B_0 + \alpha_1\, I && \alpha_2 = -\mathrm{tr}(A\, B_1)/2 \\
&\ldots && \ldots \\
&B_i = A\, B_{i-1} + \alpha_i\, I && \alpha_{i+1} = -\mathrm{tr}(A\, B_i)/(i+1) \\
&\ldots && \ldots \\
&B_{n-1} = A\, B_{n-2} + \alpha_{n-1}\, I && \alpha_n = -\mathrm{tr}(A\, B_{n-1})/n \\
&0 = A\, B_{n-1} + \alpha_n\, I &&
\end{aligned} \tag{LA.3.6}$$

Last relation is useful to check the numerical accuracy of the whole procedure.

The minimal polynomial of a matrix can be computed on the basis of the following theorem.

Theorem LA.3.2 – The minimal polynomial of A is given by

$$m(\lambda) = \frac{\det(\lambda\, I - A)}{b(\lambda)}$$

where $b(\lambda)$ is the greatest monic common divisor of the minors of order $n - 1$ of $(\lambda\; I - A)$, i.e. of the entries of $\mathrm{adj}(\lambda\; I - A)$.

The theorem which follows introduces the Singular Value Decomposition (S.V.D.), one of the most important matrix decompositions in linear algebra.

Theorem LA.3.3 (S.V.D.) – Any $(m \times n)$ real matrix A can be decomposed as

$$A = U\, \Sigma\, V^T \qquad \text{(LA.3.7)}$$

where U and V are $(m \times m)$ and $(n \times n)$ orthogonal matrices and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$ is an $(m \times n)$ diagonal matrix with $p = \min(m, n)$ and $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_p \geq 0$.

The diagonal elements σ_i are called *singular values* of A and expression (LA.3.7) is called the singular value decomposition of A.

Remark LA.3.1 – If A has rank $r > 0$, then A has exactly r strictly positive singular values, so that $\sigma_r > 0$ and $\sigma_{r+1} = \ldots = \sigma_p = 0$. If A has full rank, all its singular values are nonzero. Morover the singular values of A are the square roots of the eigenvalues of $A^T A$ when $m \geq n$ and of $A A^T$ when $m < n$. If A is symmetric, its singular values are the absolute values of its eigenvalues.

The singular value decomposition of a matrix allows an easy computation of its pseudoinverse as stated by the following Lemma.

Lemma LA.3.1 – Let A be a real $m \times n$ matrix with rank r. Then its pseudoinverse is given by

$$A^+ = V\, \Sigma^+\, U^T \qquad \text{(LA.3.8)}$$

where

$$\Sigma^+ = \begin{bmatrix} \Sigma_r^{-1} & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix}, \qquad \text{(LA.3.9)}$$

$$\Sigma_r^{-1} = \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_r \end{bmatrix}. \qquad \text{(LA.3.10)}$$

Of course when $m = n = \rho(A)$, $A^+ = A^{-1}$.