ST

System Theory

ST.1 POLYNOMIAL INPUT–OUTPUT MODELS

State space models constitute the most complete description of the internal and external behavior of dynamical systems. These models are also very useful in partitioning systems with reference to their reachability and observability properties (Kalman decomposition) to extract, when useful, the relevant part(s).

In other applications like realization of input–output sequences and identification, it is preferable to rely on models that constitute direct links between the input and the output i.e. between the measurable attributes of the system. A class of descriptions of this kind is given by polynomial input–output models that can be deduced from generic (non canonical) MIMO state space models as follows.

Consider the generic discrete–time state space model

$$x(t+1) = A\,x(t) + B\,u(t) \tag{ST.1.1a}$$
$$y(t) = C\,x(t) + D\,u(t) \tag{ST.1.1b}$$

and denote with

$$m(\lambda) = \lambda^{\mu} + a_{\mu}\,\lambda^{\mu-1} + \ldots + a_2\,\lambda + a_1 \tag{ST.1.2}$$

the minimal polynomial of A. Applying μ times the unitary advance operator z to equation (ST.1.1b) we obtain the following relations

$$\begin{aligned}
y(t) &= C\,x(t) + D\,u(t)\\
z\,y(t) &= CA\,x(t) + CB\,u(t) + D\,z\,u(t)\\
z^2\,y(t) &= CA^2\,x(t) + CAB\,u(t) + CB\,z\,u(t) + D\,z^2\,u(t)\\
&\ldots\\
z^{\mu}\,y(t) &= CA^{\mu}\,x(t) + CA^{\mu-1}B\,u(t) + \ldots + CB\,z^{\mu-1}\,u(t) + D\,z^{\mu}\,u(t).
\end{aligned} \tag{ST.1.3}$$

Summing these relations multiplied by $a_1,\ a_2,\ \ldots,a_{\mu},\ 1$ and observing that

$$C\big[\,a_1 I + a_2 A + \ldots + a_{\mu} A^{\mu-1} + A^{\mu}\,\big]x(t) = C\,m(A)\,x(t) = 0 \tag{ST.1.4}$$

we obtain

$$m(z)\,y(t) = \big[\,(a_2CB + a_3CAB + a_3CB\,z + \ldots + CB\,z^{\mu-1}) + m(z)\,D\,\big]u(t) \quad \text{(ST.1.5)}$$

which constitutes a polynomial input–output relation of the type

$$Q(z)\,y(t) = P(z)\,u(t) \quad \text{(ST.1.6)}$$

where

$$Q(z) = m(z)I \quad \text{(ST.1.7a)}$$

$$P(z) = \big[\,(a_2CB + a_3CAB + \ldots + CB\,z^{\mu-1}) + m(z)\,D\,\big]. \quad \text{(ST.1.7b)}$$

Another way for introducing and deducing input–output polynomial models relies on the decomposition of the transfer matrix

$$G(z) = C\big(zI - A\big)^{-1}B + D \quad \text{(ST.1.8)}$$

into $(m \times m)$ and $(m \times r)$ polynomial matrices $Q(z)$ and $P(z)$ such that

$$G(z) = Q(z)^{-1}P(z). \quad \text{(ST.1.9)}$$

Because of this decomposition, polynomial pairs $(Q(z), P(z))$ are also called Matrix Fraction Descriptions (MFD) of dynamical systems.

Remark ST.1.1 – MFD models (ST.1.6) can describe only the observable parts of dynamical systems. This property can be easily proved considering a Kalman decomposition of the state space model (ST.1.1) and observing that the terms CB, $CAB, \ldots,$ $CA^{\mu-1}B$ that define $P(z)$ are not influenced, because of the structure of C in the Kalman decomposition, by the non observable parts of the system.

Remark ST.1.2 – MFD models (ST.1.6) describe non completely reachable systems when the Greatest Left Common Divisor (GLCD) of $Q(z)$ and $P(z)$ is a non unimodular matrix $M(z)$. The completely reachable and observable part of the system can be extracted from a generic MFD model computing the GLCD of $Q(z)$ and $P(z)$ and extracting from the decomposition

$$M(z)\,Q'(z)\,y(t) = M(z)\,P'(z)\,u(t) \quad \text{(ST.1.10)}$$

the pair $(Q'(z), P'(z))$. Of course the non reachable part (if any) is automatically cancelled from the transfer function since

$$G(z) = Q(z)^{-1}P(z) = Q'(z)^{-1}M(z)^{-1}M(z)P'(z) = Q'(z)^{-1}P'(z). \quad \text{(ST.1.11)}$$