ID3

ARX Identification

3.6 RECURSIVE LEAST SQUARES

Previous results on the asymptotic absence of bias of least squares estimates underline the opportunity of using input–output sequences as long as possible. In many processes, inputs and outputs are sampled in normal operating conditions, over extended periods of time; in these cases it can be necessary to obtain an initial estimate of the model from a limited set of data updating this estimate every time that new measures become available. Updating can be performed either by applying repeatedly (3.3.12) to the whole set of data or updating previous estimates. This last choice can be particularly advantageous when sampling intervals are comparable with model estimation times and/or a single computing resource must update simultaneously many models.

To obtain a recursive version of least squares, consider estimate (3.3.12) of θ°, obtained at time $t = L$ using the whole set of available samples; we will underline this fact rewriting (3.3.12) according to the following notation

$$\theta^\circ(t) = S(t)^{-1} H^T(t)\, y^\circ(t). \tag{3.6.1}$$

We can write, at time $t-1$, the analogous expression

$$\theta^\circ(t-1) = S(t-1)^{-1} H^T(t-1)\, y^\circ(t-1) \tag{3.6.2}$$

where, according to (3.3.6), (3.3.7) and (3.3.8),

$$H(t-1) = \begin{bmatrix} y(1) & \dots & y(n) & u(1) & \dots & u(n) \\ \vdots & & \vdots & & & \\ y(t-n-1) & \dots & y(t-2) & u(t-n-1) & \dots & u(t-2) \end{bmatrix}. \tag{3.6.3}$$

It is now easy to verify that

$$S(t) = H^T(t) H(t) = H^T(t-1) H(t-1) + h^T(t)\, h(t) = S(t-1) + h^T(t)\, h(t) \tag{3.6.4}$$

where

$$h(t) = \left[\, y(t-n) \ldots y(t-1)\, u(t-n) \ldots u(t-1) \,\right] \tag{3.6.5}$$

denotes the last row of $H(t)$ i.e. the update of $H(t-1)$. Vector $y^\circ(t-1)$ appearing in relation (3.6.2) is given by

$$y^\circ(t-1) = \left[\, y(n+1) \; \ldots \; y(t-1) \,\right]^T \tag{3.6.6}$$

so that

$$H^T(t)\, y^\circ(t) = H^T(t-1)\, y^\circ(t-1) + h^T(t)\, y(t). \tag{3.6.7}$$

Deducing $S(t-1)$ and $H^T(t-1)\, y^\circ(t-1)$ from (3.6.4) and (3.6.7) and substituting their expressions in (3.6.2), we obtain

$$\theta^\circ(t-1) = \left[\, S(t) - h^T(t)\, h(t) \,\right]^{-1} \left[\, H^T(t)\, y^\circ(t) - h^T(t)\, y(t) \,\right]; \tag{3.6.8}$$

premultiplying (3.6.8) by $S(t)^{-1}[S(t) - h^T(t)h(t)]$ we obtain finally the following relation

$$\begin{aligned} \theta^\circ(t-1) &= S(t)^{-1} h^T(t)\, h(t)\, \theta^\circ(t-1) + S(t)^{-1} H^T(t)\, y^\circ(t) - S(t)^{-1} h^T(t)\, y(t) \\ &= \theta^\circ(t) - S(t)^{-1} h^T(t) \left[\, y(t) - h(t)\, \theta^\circ(t-1) \,\right]. \end{aligned} \tag{3.6.9}$$

Observing now that the prevision $y(t|t-1)$ is given by $h(t)\, \theta^\circ(t-1)$, relation (3.6.9) can be written as

$$\theta^\circ(t) = \theta^\circ(t-1) + K(t)\, \varepsilon(t) \tag{3.6.10}$$

where the gain matrix $K(t)$ is given by

$$K(t) = S(t)^{-1} h^T(t) \tag{3.6.11}$$

and $\varepsilon(t) = y(t) - y(t|t-1)$ denotes the difference between the observed output at time t and its prevision at time $t-1$ by a predictor parameterized by $\theta^\circ(t-1)$. The parameter vector is not updated when the prediction equals the observed output.

Previous expressions allow a recursive implementation of least squares but are not suitable for very long sequences because it can be expected that $S(t)$ will diverge for $t \to \infty$. To avoid numerical problems it is possible to rely on the update of

$$R(t) = \frac{S(t)}{N} \tag{3.6.12}$$

instead of $S(t)$; $R(t)$ can be assumed converging for $t \to \infty$. Dividing by N both sides of relation (3.6.4) we get

$$R(t) = \frac{S(t)}{N} = \frac{N-1}{N} \frac{S(t-1)}{N-1} + \frac{h^T(t)\, h(t)}{N} = \frac{N-1}{N} R(t-1) + \frac{h^T(t)\, h(t)}{N}. \tag{3.6.13}$$

The least squares algorithm can then be implemented using relations (3.6.13) and

$$K(t) = \frac{R(t)^{-1} h^T(t)}{N} \tag{3.6.14}$$

instead of (3.6.4) and (3.6.11). An efficient implementation of recursive least squares must avoid recomputing, at every step, the inverse of $R(t)$; using a well–known matrix inversion lemma , it is possible to express $R(t)^{-1}$ as

$$\begin{aligned} R(t)^{-1} = &\frac{N\,R(t-1)^{-1}}{N-1} \\ &- \frac{N\,R(t-1)^{-1}\,h^T(t)}{N-1}\left[N + \frac{N\,h(t)\,R(t-1)^{-1}\,h^T(t)}{N-1}\right]^{-1}\frac{N\,h(t)\,R(t-1)^{-1}}{N-1} \end{aligned} \tag{3.6.15}$$

or also, by simple steps,

$$R(t)^{-1} = \frac{N\,R(t-1)^{-1}}{N-1}\left[I - \frac{h^T(t)\,h(t)\,R(t-1)^{-1}}{N-1+h(t)\,R(t-1)^{-1}h^T(t)}\right] \tag{3.6.16}$$

which requires, being $h(t)R(t-1)^{-1}h^T(t)$ a scalar, a limited number of operations for updating $R(t)^{-1}$. An on–line implementation of least squares can thus be obtained performing an initial parameter estimate

$$\theta^\circ(t_0) = S(t_0)^{-1} H^T(t_0)\,y^\circ(t_0) = \frac{R(t_0)^{-1}}{N_0}\,H^T(t_0)\,y^\circ(t_0) \tag{3.6.17}$$

and subsequent updates by means of (3.6.10) and (3.6.14), relying on (3.6.16), to update $R(t)^{-1}$.