ID3

ARX Identification

3.22 STATE SPACE ARX MODELS

Despite the introduction of ARX models as input–output descriptions before the development of the state space approach, it is of some interest to describe their stochastic environment also in the state space. The tools necessary for this analysis are offered by realization theory and we will directly consider the more general multivariable case.

3.22.1 Multivariable ARX state space models

Consider the minimally parameterized model (3.17.3) with $u(t)$ and $e(t)$ as inputs. The row degrees of $P(z)$ and $D(z)$ show that (3.17.3) describes a system that is purely dynamic with respect to $u(t)$ and non purely dynamic with respect to $e(t)$ so that all corresponding state space realizations have structures of the type

$$x(t+1) = A\,x(t) + B_u u(t) + B_e e(t) \tag{3.22.1a}$$

$$y(t) = C\,x(t) + D_e e(t). \tag{3.22.1b}$$

A and C can be immediately deduced from $Q(z)$ according to (ST.4.12), (ST.3.12) and (ST.3.13); $Q(z)$ allows to write also matrix M (ST.4.16) while $\overline{B}_u$ (ST.4.14) is directly obtained from $P(z)$. Inverting M we obtain the matrix

$$\Phi_u = M^{-1}\,\overline{B}_u \tag{3.22.2}$$

whose rows define the matrices B_u and D_u $(=0)$ acording to (ST.4.15). It is now possible to construct $\overline{B}_e$ given by

$$\overline{B}_e = \begin{bmatrix} \overline{B}_{e1} \\ \vdots \\ \overline{B}_{em} \end{bmatrix} \qquad \overline{B}_{ei} = \begin{array}{c} \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \end{bmatrix} \\ \begin{array}{ccccc} & & \uparrow & & \\ & & i & & \end{array} \end{array} \tag{3.22.3}$$

and the matrix

$$\Phi_e = M^{-1}\overline{B}_e \tag{3.22.4}$$

whose rows define B_e and D_e according to (ST.4.15). Since the entries of $\overline{B}_e$ contain only null and unitary elements in well defined positions, the entries of B_e and D_e depend only on those of M^{-1}, i.e. on the entries of A. Considering the structure of M^{-1}, given by

$$M^{-1} = [\, M^{-1} \,]_{ij} \quad (i, j = 1, \ldots, m) \tag{3.22.5a}$$

$$[\, M^{-1} \,]_{ii} = \begin{bmatrix} 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & 1 & \times \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \ldots & \times & \times \\ 1 & \times & \ldots & \times & \times \end{bmatrix}_{(\nu_i + 1 \times \nu_i + 1)} \tag{3.22.5b}$$

$$[\, M^{-1} \,]_{ij} = \begin{bmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 & \times \\ 0 & 0 & \ldots & \times & \times \\ \vdots & \vdots & & & \vdots \\ 0 & \times & \ldots & \times & \times \end{bmatrix}_{(\nu_i + 1 \times \nu_j + 1)}, \tag{3.22.5c}$$

it is easy to show that

$$D_e = Q^{*-1}_{\nu_M + 1} \tag{3.22.6}$$

where $Q^*_{\nu_M+1}$ is defined in (3.17.7a).

3.22.2 Example – Consider a canonical input–output model (3.17.3) with structure $\nu = (2, 1)$, defined by the following polynomial matrices

$$Q(z) = \begin{bmatrix} z^2 - 5z + 1 & 2 \\ -z + 2 & z - 3 \end{bmatrix} \qquad P(z) = \begin{bmatrix} z - 5 \\ -1 \end{bmatrix}.$$

It follows that

$$D(z) = \begin{bmatrix} z^2 \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 5 & -2 \\ -2 & 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

M, M^{-1} and $\overline{B}_e$ are

$$M = \begin{bmatrix} 1 & -5 & 1 & 2 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & -3 & 1 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 \\ 1 & 5 & 22 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 6 & 1 & 3 \end{bmatrix}$$

$$\overline{B}_u = \begin{bmatrix} -5 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

so that

$$\Phi_u = M^{-1}\,\overline{B}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad D_u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have also

$$\overline{B}_e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\Phi_e = M^{-1}\,\overline{B}_e = \begin{bmatrix} 1 & 0 \\ 5 & 0 \\ 22 & -2 \\ 1 & 1 \\ 6 & 3 \end{bmatrix} \qquad B_e = \begin{bmatrix} 5 & 0 \\ 22 & -2 \\ 6 & 3 \end{bmatrix} \qquad D_e = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It can be immediately verified that

$$Q^*_{\nu_M+1} = Q_3 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

and that $D_e = Q^{*-1}_{\nu_M+1}$.

3.22.3 MISO ARX state space models

The MISO case can be deduced from the MIMO one taking $\nu = (n)$ and

$$D(z) = \left[\, z^n \,\right]. \tag{3.22.7}$$

It follows that

$$\overline{B}_e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.22.8}$$

so that Φ_e coincides with the last column of M^{-1} and

$$D_e = \left[\, 1 \,\right]. \tag{3.22.9}$$

The corresponding state space model has the following structure

$$x(t+1) = A\,x(t) + B_u u(t) + B_e e(t) \tag{3.22.10a}$$
$$y(t) = C\,x(t) + e(t). \tag{3.22.10b}$$

where, as in the MIMO case, $B_e = B_e(A)$.