ST

System Theory

ST.3 STATE SPACE CANONICAL FORMS

Consider the linear discrete–time multivariable system

$$x(t+1) = A\,x(t) + B\,u(t) \tag{ST.3.1a}$$

$$y(t) = C\,x(t) + D\,u(t) \tag{ST.3.1b}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^r$ and $y \in \mathcal{R}^m$, assuming its complete observability, i.e.

$$\text{rank}\begin{bmatrix} C \\ CA \\ \cdots \\ CA^{(n-1)} \end{bmatrix} = n. \tag{ST.3.2}$$

We will also assume, without loss of generality, that

$$\text{rank}\, C = m \tag{ST.3.3}$$

i.e. that no output is a linear combination of remaining ones; this restriction has no conceptual relevance and is introduced only to avoid unnecessarily complex notations. Consider now the sequences of vectors

$$\begin{array}{llll} c_1^T & A^T c_1^T & A^{T^2} c_1^T & \dots \\ c_2^T & A^T c_2^T & A^{T^2} c_2^T & \dots \\ \vdots & & & \\ c_m^T & A^T c_m^T & A^{T^2} c_m^T & \dots \end{array} \tag{ST.3.4}$$

given by the trasposed rows of the observability matrix (ST.3.2) and select them according to the following order

$$c_1^T \quad c_2^T \quad \dots \quad c_m^T \quad A^T c_1^T \quad A^T c_2^T \quad \dots \quad A^T c_m^T \quad \dots \tag{ST.3.5}$$

testing the linear dependence of every vector on previous ones. As soon as a dependent vector ${A^T}^{\nu_i} c_i^T$, is found, it is possible to write the relation

$$ {A^T}^{\nu_i} c_i^T = \sum_{j=1}^{m} \sum_{k=1}^{\nu_{ij}} \alpha_{ijk} \, {A^T}^{(k-1)} c_j^T \tag{ST.3.6} $$

where, because of selection order (ST.3.5),

$$ \nu_{ij} = \nu_i \qquad \text{for } i = j \tag{ST.3.7a} $$
$$ \nu_{ij} = \min(\nu_i + 1, \nu_j) \qquad \text{for } i > j \tag{ST.3.7b} $$
$$ \nu_{ij} = \min(\nu_i, \nu_j) \qquad \text{for } i < j. \tag{ST.3.7c} $$

All subsequent vectors ${A^T}^k c_i^T$ $(k > \nu_i)$ belonging to the same chain will be necessarily dependent and their test is unnecessary. Once dependent vectors belonging to all m chains (ST.3.4) have been found, a total of

$$ n = \nu_1 + \nu_2 + \ldots + \nu_m \tag{ST.3.8} $$

independent vectors have been selected, because of the complete observability assumption. The m linear dependence relations (ST.3.6) are described by

$$ \ell = \sum_{i=1}^{m} \sum_{j=1}^{m} \nu_{ij} \tag{ST.3.9} $$

scalars α_{ijk}. Refer now model (ST.3.1) to the new state space basis given by

$$ T = \begin{bmatrix} c_1 \\ \vdots \\ c_1 A^{(\nu_1 - 1)} \\ \vdots \\ c_m \\ \vdots \\ c_m A^{(\nu_m - 1)} \end{bmatrix}^{-1} . \tag{ST.3.10} $$

Because of the structure of T and T^{-1}, given by

$$ T^{-1} = \left[\, c_1^T \ldots {A^T}^{(\nu_1 - 1)} c_1^T \mid \ldots \mid c_m^T \ldots {A^T}^{(\nu_m - 1)} c_m^T \,\right] \tag{ST.3.11} $$

it is easy to show that the model that we obtain exhibits the following structure

$$ \tilde{A} = T^{-1} A T = \left[\, A_{ij} \,\right] \quad (i = 1, \ldots, m;\; j = 1, \ldots, m) \tag{ST.3.12a} $$

$$\tilde{A}_{ii} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ \alpha_{ii1} & \alpha_{ii2} & \dots & \alpha_{ii\nu_i} \end{bmatrix}_{(\nu_i \times \nu_i)} \tag{ST.3.12b}$$

$$\tilde{A}_{ij} = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \\ \alpha_{ij1} & \dots & \alpha_{ij\nu_{ij}} & 0 & \dots & 0 \end{bmatrix}_{(\nu_i \times \nu_j)} \tag{ST.3.12c}$$

$$\tilde{C} = CT = \begin{array}{c} \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \qquad\qquad \uparrow & \qquad\qquad \uparrow \\ 1 & \qquad (\nu_1 + 1) & (\nu_1 + \ldots + \nu_{m-1} + 1) \end{array} \end{array} \tag{ST.3.13}$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ \vdots \\ \tilde{B}_m \end{bmatrix} \qquad \tilde{B}_i = \begin{bmatrix} b_{i11} & \dots & b_{ir1} \\ \vdots & & \vdots \\ b_{i1\nu_i} & \dots & b_{ir\nu_i} \end{bmatrix} = \begin{bmatrix} b_{i1} \\ \vdots \\ b_{i\nu_i} \end{bmatrix} \tag{ST.3.14}$$

$$\tilde{D} = D = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} d_{i1} & \dots & d_{ir} \\ \vdots & & \vdots \\ d_{m1} & \dots & d_{mr} \end{bmatrix}. \tag{ST.3.15}$$

Remark ST.3.1 – The set of integers $\nu_1, \ldots, \nu_m$ will be called the (observability) *structure* of model (ST.3.1) and will be denoted by means of the multi–index

$$\nu = (\nu_1, \ldots, \nu_m). \tag{ST.3.16}$$

Obviously the order n of the system is given by

$$n = \sum_{i=1}^{m} \nu_i . \tag{ST.3.17}$$

Remark ST.3.2 – The procedure that has been outlined has implicitly defined a function f associating to every quadruple (A, B, C, D) the scalars $(\nu_i, \alpha_{ijk}, b_{ijk}, d_{ij})$. f constitutes a complete set of independent invariants for the equivalence relation associated with a change of basis in the state space. As a consequence, none of these scalars can be expressed as a function of remaining ones and the parameterization of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is minimal.

Remark ST.3.3 – The procedure leads to a single model $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ for every quadruple (A, B, C, D) belonging to the same equivalence class. Consequently it defines a subset of canonical forms for the equivalence relation that has been considered.

Example – Consider the system described by the quadruple

$$A = \frac{1}{12}\begin{bmatrix} 4 & 2 & 2 & 0 & 2 \\ 4 & 20 & 12 & -8 & 16 \\ -2 & -12 & -6 & -2 & -12 \\ 0 & -2 & -2 & 4 & -2 \\ -2 & -2 & 0 & 10 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{ST.3.18a}$$

$$C = \begin{bmatrix} 2 & 0 & -2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}. \tag{ST.3.18b}$$

Vectors (ST.3.4) are given by

$$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \quad \frac{1}{12}\begin{bmatrix} 10 \\ 24 \\ 14 \\ 18 \\ 28 \end{bmatrix} \quad \frac{1}{144}\begin{bmatrix} 52 \\ 240 \\ 188 \\ 132 \\ 256 \end{bmatrix} \quad \frac{1}{1728}\begin{bmatrix} 280 \\ 1872 \\ 1592 \\ 792 \\ 1936 \end{bmatrix} \quad \dots \tag{ST.3.19a}$$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad \frac{1}{12}\begin{bmatrix} 8 \\ 22 \\ 18 \\ 20 \\ 26 \end{bmatrix} \quad \frac{1}{144}\begin{bmatrix} 32 \\ 148 \\ 132 \\ 128 \\ 164 \end{bmatrix} \quad \frac{1}{1728}\begin{bmatrix} 128 \\ 856 \\ 792 \\ 704 \\ 920 \end{bmatrix} \quad \dots \tag{ST.3.19b}$$

Selecting these vectors according to (ST.3.5) and testing their independence on previous ones we find that the first dependent vector is the third vector of the second chain $(A^{T^2} c_2^T)$; this means that $\nu_2 = 2$. $(A^{T^2} c_2^T)$ is then discarded and no other vectors of the second chain are selected; the second dependent vector is $(A^{T^3} c_1^T)$ and this completes the determination of the system structure establishing that $\nu_1 = 3$. The corresponding new basis of the state space is

$$T = \frac{1}{144}\begin{bmatrix} 288 & 0 & -288 & 144 & 144 \\ 120 & 288 & 168 & 216 & 336 \\ 52 & 240 & 188 & 132 & 256 \\ 288 & 288 & 144 & 288 & 432 \\ 96 & 264 & 216 & 240 & 312 \end{bmatrix}. \tag{ST.3.20}$$

The resulting canonical model is

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.0794 & -0.5635 & 1.3095 & -0.0079 & 0.0238 \\ 0 & 0 & 0 & 0 & 1 \\ 0.0238 & -0.1190 & 0.1429 & -0.1190 & 0.6905 \end{bmatrix} \quad \text{(ST.3.21a)}$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 1.6667 & 0 \\ 3 & 1 \\ 2.1667 & 0.3333 \end{bmatrix} \quad \text{(ST.3.21b)}$$

$$\tilde{C} = CT = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \tilde{D} = D = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}. \quad \text{(ST.3.21c)}$$