SP

Stochastic Processes

SP.2 STOCHASTIC PROCESSES

A *random* or *stochastic process* is a function of time determined by the outcome ω of a random experiment

$$x(t) = x(t, \omega). \tag{SP.2.1}$$

For every ω, $x(t, \omega)$ is a function of t called a *realization* of the stochastic process. The probability that $x(t)$ assumes values in a certain range is given by its probability distribution function, as for any random variable; the dependence of the observation on time is shown explicity using the notation $\mathrm{P}(x_1, t) = \Pr(x(t) \leq x_1)$. The corresponding probability density function is

$$\mathrm{p}(x_1, t) = \frac{\mathrm{d}P(x_1, t)}{\mathrm{d}x_1}. \tag{SP.2.2}$$

If $t \in \mathcal{Z}$ then $x(t, \omega)$ is called *random sequence* or *discrete–time stochastic process*.

Definition SP.2.1 (Mean, autocorrelation and autocovariance) – The *mean* or *expected value* of a stochastic process $x(t)$ at time t is defined as

$$\bar{x}(t) = \mathrm{E}\,[\,x(t)\,] \tag{SP.2.3}$$

while its *autocorrelation* at times t_1 and t_2 is defined as

$$\mathrm{R}_x(t_1, t_2) = \mathrm{E}\left[\,x(t_1)\,x^T(t_2)\,\right]. \tag{SP.2.4}$$

The *autocovariance* of $x(t)$ is defined as

$$\mathrm{C}_x(t_1, t_2) = \mathrm{E}\left[(x(t_1) - \bar{x}(t_1))\,(x(t_2) - \bar{x}(t_2))^T\right] \tag{SP.2.5}$$

and satisfies the relation

$$\mathrm{C}_x(t_1, t_2) = \mathrm{R}_x(t_1, t_2) - \bar{x}(t_1)\bar{x}^T(t_2). \tag{SP.2.6}$$

Remark SP.2.1 – The autocorrelation and autocovariance of a scalar stochastic process $x(t)$, are denoted with the lower case notations $\mathrm{r}_x(t_1, t_2)$ and $\mathrm{c}_x(t_1, t_2)$.

Definition SP.2.2 (White noise) – A zero–mean scalar random process $x(t)$ whose autocorrelation is zero for any couple of different times is called *white noise*. In this case $r_x = q\,\delta(t_1 - t_2)$ where $\delta(t)$ denotes the impulse function.

Definition SP.2.3 (Discrete–time white noise) – The random scalar sequence $x(t)$, $t \in \mathcal{Z}$, is a (discrete–time) white noise if

$$\mathrm{r}_x(t_1, t_2) = q\,\delta_{t_1\,t_2}, \qquad t_1, t_2 \in \mathcal{Z} \tag{SP.2.7}$$

where $\delta_{t_1\,t_2}$ is the Kronecker delta.

Definition SP.2.4 (Cross correlation and cross covariance) – The *cross correlation* of two stochastic processes $x(t)$ and $y(t)$ at times t_1 and t_2 is defined as

$$\mathrm{R}_{xy}(t_1, t_2) = \mathrm{E}\left[\, x(t_1)\, y^T(t_2)\,\right] \tag{SP.2.8}$$

and the *cross covariance* as

$$\begin{aligned} \mathrm{C}_{xy}(t_1, t_2) &= \mathrm{E}\left[(x(t_1) - \bar{x}(t_1))\,(y(t_2) - \bar{y}(t_2))^T\,\right] \\ &= \mathrm{R}_{xy}(t_1, t_2) - \bar{x}(t_1)\bar{y}^T(t_2) = \mathrm{C}_{yx}^T(t_2, t_1). \end{aligned} \tag{SP.2.9}$$

Two stochastic processes are defined as *uncorrelated* if $\mathrm{R}_{xy}(t_1, t_2) = \mathrm{E}[x(t_1)]\,\mathrm{E}[y^T(t_2)]\ \forall\, t_1, t_2$.

Previous definitions refer to specific values of time; stochastic processes are defined stationary when their statistical properties are time–invariant.

Definition SP.2.5 (Wide sense stationary stochastic processes) – A stochastic process $x(t, \omega)$ is called *(wide sense) stationary* if its expectation $\bar{x}(t) = \mathrm{E}[x(t)]$ is independent of t and its autocorrelation $\mathrm{R}_x(t_1, t_2)$ depends only on $\tau = t_1 - t_2$. The autocorrelation is denoted, in this case, as $\mathrm{R}_x(\tau)$.

Remark SP.2.2 – If $x(t)$ and $y(t)$ are stationary stochastic processes with expected values $\bar{x}$ and $\bar{y}$ and $\tau = t_1 - t_2$, then $\mathrm{R}_{xy}(t_1, t_2) = \mathrm{R}_{xy}(\tau)$ and

$$\begin{aligned} \mathrm{C}_{xy}(t_1, t_2) &= \mathrm{R}_{xy}(t_1, t_2) - \bar{x}(t_1)\bar{y}^T(t_2) \\ &= \mathrm{R}_{xy}(\tau) - \bar{x}\,\bar{y} = \mathrm{C}_{xy}(\tau) = \mathrm{C}_{yx}^T(-\tau). \end{aligned} \tag{SP.2.10}$$

Assuming, in previous relations, $x = y$, we obtain $\mathrm{C}_x(t_1, t_2) = \mathrm{C}_x(\tau) = \mathrm{C}_x^T(-\tau)$.