

LA

Linear

Algebra



LA.1 EQUIVALENCE AND INVARIANTS

Definition LA.1.1 – Denote with X a set and with E an equivalence relation defined on X . Denote then with S a second set and with $f : X \rightarrow S$ a function. If x' and x'' are two elements of X and f is such that $x' E x''$ implies $f(x') = f(x'')$, f is called an *invariant* for E . Moreover if $f(x') = f(x'')$ implies $x' E x''$, f is called a *complete invariant* for E .

If f is a complete invariant for E , then all elements of X belonging to the same equivalence class have the same image in f ; moreover, these classes coincide exactly with the inverse images in f of the elements of the image (or range) of f . There exists, therefore, a bijection between the quotient set X/E and the image of a complete invariant for E .

Definition LA.1.2 – A set of invariants $f_1, \dots, f_n, f_i : X \rightarrow S_i$ for E is called a *complete set of invariants* for E if the function $f = (f_1, \dots, f_n) : X \rightarrow S_1 \times \dots \times S_n$ defined by $x \rightarrow (f_1(x), \dots, f_n(x))$ is a complete invariant for E .

Definition LA.1.3 – A set of invariants for $E, f_1, \dots, f_n, f_i : X \rightarrow S_i$, will be called *independent* if the associated invariant $f = (f_1, \dots, f_n) : X \rightarrow S_1 \times \dots \times S_n$ is surjective.

This condition implies that no invariant f_i can be expressed as a function of the others. This last condition, however, is weaker than the definition of independence that has been given. A complete set of independent invariants for E is also called a *basis* for E on X .

Lemma LA.1.1 – Let $f : X \rightarrow S$ be a complete surjective invariant for E . Then every other invariant for E can be uniquely computed from f .

Proof: Let $f : X \rightarrow S$ and $g : X \rightarrow T$ be, respectively, a complete surjective invariant and a generic invariant for E . Commutativity in the diagram of Fig. [LA.1.1](#)

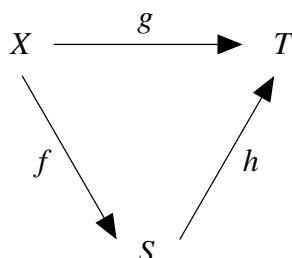


Figure LA.1.1

can be obtained if and only if for every element s of S the function h is defined as $h(s) = g(x)$ where x is any element of X such that $f(x) = s$. Since f is complete and surjective and g is an invariant, h is well defined for all elements of S .

Corollary LA.1.1 – Let $f : X \rightarrow S$ be a complete set of independent invariants for E . Then every other invariant for E can be uniquely computed from f .

Property LA.1.1 – Let $f : X \rightarrow S$ be a complete set of independent invariants for E . If $g : S \rightarrow R$ is a bijection, then $h = g \cdot f : X \rightarrow R$ is a complete set of independent invariants for E .

CANONICAL FORMS FOR EQUIVALENCE RELATIONS

Definition LA.1.4 – Let E be an equivalence relation on X . A subset C of X is called a set of *canonical forms* for E if every $x \in X$ is equivalent under E to one and only one element of C ; this element is *the* canonical form of x . The function $g : X \rightarrow C$ thus defined is therefore a complete invariant for E . Obviously g can be assumed surjective without loss of generality.

COMPLETE SETS OF INDEPENDENT INVARIANTS AND CANONICAL FORMS

Let $f : X \rightarrow S$ be a complete set of independent invariants and C a set of canonical forms for E . Then (Corollary LA.1.1) there exists a unique function $h : S \rightarrow C$ such that $g = h \cdot f$. Since g is complete, h is a bijection. Moreover if $i : C \rightarrow X$ is the injection $i(c) = c$, it follows that $h^{-1} = f \cdot i$. The following theorem has thus been proved.

Theorem LA.1.1 – Let C be a set of canonical forms for an equivalence relation E on X and f a complete set of independent invariants for E . Then there exists a unique bijection between C and the image of f .

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