ID3

ARX Identification

3.12 EFFICIENCY OF LEAST SQUARES ESTIMATES

A non biased estimate is defined as efficient when its covariance matrix equals Cramér–Rao lower bound. It will be shown now that least squares estimates of ARX models are efficient when $e(\cdot)$ is Gaussian. Consider relation

$$y^\circ = H\,\theta + e^\circ \tag{3.12.1}$$

where y°, H and e° are defined by (3.3.4), (3.3.8) and (3.5.3). The probability density function (pdf) of observations y°, conditioned by θ and by the variance, σ_e^2, of $e(t)$, is

$$p(y^\circ;\theta,\sigma_e^2) = \frac{1}{\sqrt{(2\pi\sigma_e^2)^N}}\ \exp\left[-\frac{1}{2\sigma_e^2}\left(y^\circ - H\,\theta\right)^T\left(y^\circ - H\,\theta\right)\right]; \tag{3.12.2}$$

it follows that

$$\log p(y^\circ;\theta,\sigma_e^2) = -\frac{1}{2\sigma_e^2}\left(y^\circ - H\,\theta\right)^T\left(y^\circ - H\,\theta\right) - \frac{N}{2}\log\sigma_e^2 - \frac{N}{2}\log 2\pi. \tag{3.12.3}$$

The first and second order derivatives of $\log p(y^\circ;\theta,\sigma_e^2)$ are given by

$$\frac{\partial}{\partial\theta}\log p(y^\circ;\theta,\sigma_e^2) = \frac{1}{\sigma_e^2}\,H^T\left(y^\circ - H\,\theta\right) \tag{3.12.4}$$

$$\frac{\partial}{\partial\sigma_e^2}\log p(y^\circ;\theta,\sigma_e^2) = \frac{1}{2\sigma_e^4}\left(y^\circ - H\,\theta\right)^T\left(y^\circ - H\,\theta\right) - \frac{N}{2\sigma_e^2} \tag{3.12.5}$$

$$\frac{\partial}{\partial\sigma_e^2}\frac{\partial}{\partial\theta}\log p(y^\circ;\theta,\sigma_e^2) = -\frac{1}{\sigma_e^4}\,H^T\left(y^\circ - H\,\theta\right) \tag{3.12.6}$$

$$\frac{\partial^2}{\partial\theta^2}\log p(y^\circ;\theta,\sigma_e^2) = -\frac{H^T H}{\sigma_e^2} \tag{3.12.7}$$

$$\frac{\partial^2}{\partial\,(\sigma_e^2)^2}\log p(y^\circ;\theta,\sigma_e^2) = -\frac{1}{\sigma_e^6}\left(y^\circ - H\,\theta\right)^T\left(y^\circ - H\,\theta\right) + \frac{N}{2\sigma_e^4}; \tag{3.12.8}$$

since $\mathrm{E}[H^T e^\circ] = 0$ and $\mathrm{E}[(y^\circ - H\theta)^T (y^\circ - H\theta)] = N\sigma_e^2$,

$$F = \mathrm{E}\begin{bmatrix} -\dfrac{\partial^2}{\partial\theta^2}\log p(y^\circ;\theta,\sigma_e^2) & -\dfrac{\partial}{\partial\theta}\dfrac{\partial}{\partial\sigma_e^2}\log p(y^\circ;\theta,\sigma_e^2) \\ -\dfrac{\partial}{\partial\sigma_e^2}\dfrac{\partial}{\partial\theta}\log p(y^\circ;\theta,\sigma_e^2) & -\dfrac{\partial^2}{\partial(\sigma_e^2)^2}\log p(y^\circ;\theta,\sigma_e^2) \end{bmatrix} = \begin{bmatrix} \dfrac{\mathrm{E}[H^T H]}{\sigma_e^2} & 0 \\ 0 & \dfrac{N}{2\sigma_e^4} \end{bmatrix}. \tag{3.12.9}$$

It follows that non biased estimates of ARX processes with Gaussian equation errors satisfy the following relations

$$\operatorname{cov}\theta^\circ \geq \sigma_e^2\,\mathrm{E}\big[\big(H^T H\big)^{-1}\big], \qquad \operatorname{var}\hat{\sigma}_e^2 \geq \frac{2\sigma_e^4}{N}. \tag{3.12.10}$$

(3.8.4) allows then to establish that least squares estimates are efficient. It is also possible to show, on the basis of (3.12.10), the efficiency of estimate (3.10.5) of σ_e^2.

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