ID6

ARMAX Identification

6.17 MULTIVARIABLE ARMAX MODELS

Multivariable ARMAX models can be introduced, as has been done for ARX and AR models, extending MISO models; to follow this way we would make reference to relations (6.1.11) or (6.1.12). It is however more convenient to rely on the well known equivalence between the stochastic environment of input–output ARMAX models and that of Kalman filtering, starting from state space models with the following structure

$$x(t+1) = A\,x(t) + B\,u(t) + v_x(t) \tag{6.17.1a}$$

$$y(t) = Cx(t) + v_y(t) \tag{6.17.1b}$$

where $v_x(t) \in \mathcal{R}^n$ and $v_y(t) \in \mathcal{R}^m$ are independent white processes with covariance matrices Q_x and Q_y. Model (6.17.1) can be written in the innovations form

$$x(t+1) = A\,x(t) + B\,u(t) + K\,w(t) \tag{6.17.2a}$$

$$y(t) = Cx(t) + w(t) \tag{6.17.2b}$$

where K is the gain of the Kalman filter asociated with model (6.17.1) and $w(t) \in \mathcal{R}^m$ is the vector of its innovations. Consider now the auxiliary output

$$y^*(t) = Cx(t) \tag{6.17.3}$$

and the MFD model linking, according to (ST.4.9) – (ST.4.16), $u(t)$ and $w(t)$ to $y^*(t)$

$$Q(z)\,y^*(t) = P(z)\,u(t) + R(z)\,w(t) \tag{6.17.4}$$

where $Q(z)$, $P(z)$ and $R(z)$ are the polynomial matrices

$$Q(z) = \left[\, q_{ij}(z) \,\right] \ \ (i,\, j = 1, \ldots, m) \tag{6.17.5a}$$

$$q_{ii}(z) = z^{\nu_i} - \alpha_{ii\nu_i}\, z^{(\nu_i - 1)} - \ldots - \alpha_{ii1} \tag{6.17.5b}$$

$$q_{ij}(z) = -\alpha_{ij\nu_{ij}}\, z^{(\nu_{ij}-1)} - \ldots - \alpha_{ij2}\, z - \alpha_{ij1} \tag{6.17.5c}$$

$$P(z) = \big[\, p_{ij}(z) \,\big] \;\; (i = 1, \ldots, m, \;\; j = 1, \ldots, p) \tag{6.17.6a}$$

$$p_{ij}(z) = \beta_{ij\nu_i}\, z^{(\nu_i-1)} + \ldots + \beta_{ij2}\, z + \beta_{ij1} \tag{6.17.6b}$$

$$R(z) = \big[\, r_{ij}(z) \,\big] \;\; (i, j = 1, \ldots, m) \tag{6.17.7a}$$

$$r_{ij}(z) = \gamma_{ij\nu_i}\, z^{(\nu_i-1)} + \ldots + \gamma_{ij2}\, z + \gamma_{ij1}. \tag{6.17.7b}$$

The complete ARMAX model can now be deduced substituting $y^*(t)$ with $y(t) - w(t)$ in (6.17.4); we obtain

$$Q(z)\, y(t) = P(z)\, u(t) + \big(Q(z) + R(z)\big) w(t) = P(z)\, u(t) + S(z)\, w(t). \tag{6.17.8}$$

The number of parameters, ℓ, in $Q(z)$, $P(z)$ and $R(z)$ is

$$\ell = \sum_{i=1}^{m} \sum_{j=1}^{m} \nu_{ij} + n(m+p) \;\; \text{where} \;\; \nu_{ii} = \nu_i; \tag{6.17.9}$$

these parameters will be denoted with the vector $\theta^\circ \in \mathcal{R}^\ell$

$$\begin{aligned} \theta^\circ = \big[\, & \alpha_{111} \ldots \alpha_{11\nu_1} | \ldots | \alpha_{1m1} \ldots \alpha_{1m\nu_{1m}} | \ldots | \alpha_{mm1} \ldots \alpha_{mm\nu_m} | \\ & | \beta_{111} \ldots \beta_{11\nu_1} | \ldots | \beta_{1p1} \ldots \beta_{1p\nu_1} | \ldots | \beta_{mp1} \ldots \beta_{mp\nu_m} | \\ & | \gamma_{111} \ldots \gamma_{11\nu_1} | \ldots | \gamma_{1m1} \ldots \gamma_{1m\nu_1} | \ldots | \gamma_{mm1} \ldots \gamma_{mm\nu_m} \big]^T. \end{aligned} \tag{6.17.10}$$

Remark 6.17.1 – The MFD ARMAX model (6.17.8) could have been directly obtained from (6.17.2) without introducing the auxiliary purely dynamic model with output $y^*(t)$. The reasons behind the procedure that has been followed concern the identifiability of these models that is assured only for the triple $(Q(z), P(z), R(z))$, as shown in the following.

Identifiability of multivariable ARMAX models – The triple $(Q(z), P(z), R(z))$ defines univocally the ARMAX model $(Q(z), P(z), S(z))$ since $S(z) = Q(z) + R(z)$; moreover its parameterization θ° is minimal and this assures the identifiability of this model. It is important to observe that the parameterization $\overline{\theta}^\circ \in \mathcal{R}^\iota$ of $(Q(z), P(z), S(z))$ is not, in general, minimal and not even identifiable. The proof can be easily obtained observing that, because of the relations between the degrees of the entries of $Q(z)$ described by (ST.4.18), some coefficients of the polynomials of $S(z)$ can be equal to coefficients of polynomials in $Q(z)$; this happens when there exist integers i and j, $i < j$ such that $\nu_i > \nu_j$. These considerations show that the triple $(Q(z), P(z), S(z))$, differently from $(Q(z), P(z), R(z))$, is not suitable for identification purposes because of the lack of independence among its coefficients that would lead to estimate models non compatible with the ARMAX structure (6.17.2). The example that follows illustrates this aspect.

EXAMPLE 6.6 – Consider an ARMAX model (6.17.4) with $m = 2$, $p = 1$ and $\nu = (2, 1)$; the matrices $Q(z)$, $P(z)$ and $R(z)$ will be of the following type

$$Q(z) = \begin{bmatrix} z^2 - \alpha_{112}\, z - \alpha_{111} & -\alpha_{121} \\ -\alpha_{212}\, z - \alpha_{211} & z - \alpha_{221} \end{bmatrix}$$

$$P(z) = \begin{bmatrix} \beta_{112}\, z + \beta_{111} \\ \beta_{211} \end{bmatrix} \qquad R(z) = \begin{bmatrix} \gamma_{112}\, z + \gamma_{111} & \gamma_{122}\, z + \gamma_{121} \\ \gamma_{211} & \gamma_{221} \end{bmatrix}$$

The total number of significant parameters in this triple is 15; consider now the matrix $S(z) = Q(z) + R(z)$ given by

$$S(z) = \begin{bmatrix} z^2 + (\gamma_{112} - \alpha_{112})\, z + (\gamma_{111} - \alpha_{111}) & \gamma_{122}\, z + (\gamma_{121} - \alpha_{121}) \\ -\alpha_{212}\, z + (\gamma_{211} - \alpha_{211}) & z + (\gamma_{221} - \alpha_{221}) \end{bmatrix}.$$

The total number of significant parameters in $(Q(z), P(z), S(z))$ is 16; they are, however, not independent since α_{212} appears in both $Q(z)$ and $S(z)$. Denote now by $(\hat{Q}(z), \hat{P}(z), \hat{S}(z))$ a triple estimated from noisy data; since no common parameter will be observed in $\hat{Q}(z)$ and $\hat{S}(z)$, $\hat{R}(z) = \hat{S}(z) - \hat{Q}(z)$ will exhibit a structure of the following type

$$\hat{R}(z) = \begin{bmatrix} \hat{\gamma}_{112}\, z + \hat{\gamma}_{111} & \hat{\gamma}_{122}\, z + \hat{\gamma}_{121} \\ \hat{\gamma}_{212}\, z + \hat{\gamma}_{211} & \hat{\gamma}_{221} \end{bmatrix}$$

corresponding to the presence of an algebraic link between $y^*(t)$ and $w(t)$

$$y^*(t) = Cx(t) + D\, w(t)$$

and, eventually, to an observation equation given by

$$y(t) = Cx(t) + (I + D)\, w(t).$$

The consequence of the non minimal parametrization of $(Q(z), P(z), S(z))$ results thus in non identifiability conditions since approximations in the estimated parameters lead to state space models non congruent with the ARMAX scheme.