ID4

AR Identification

4.11 YULE–WALKER EQUATIONS FOR MULTIVARIABLE AR MODELS

Yule–Walker equations can be deduced for MIMO models proceeding as already done in Module 4.5 for MISO models. It is more convenient, from the notational standpoint, to start from the forward model (4.9.1) rewriting it as follows

$$y_i(t+\nu_i) - \sum_{j=1}^{m} \sum_{k=1}^{\nu_{ij}} \alpha_{ijk}\, y_j(t+k-1) = e_i(t+\nu_i). \tag{4.11.1}$$

Consider then the quantity

$$\mathrm{E}\left[\, y_h(t+s)\, y_i(t+\nu_i) - \sum_{j=1}^{m} \sum_{k=1}^{\nu_{ij}} \alpha_{ijk}\, y_h(t+s)\, y_j(t+k-1) \,\right] = \mathrm{E}\left[\, y_h(t+s)\, e_i(t+\nu_i) \,\right] \tag{4.11.2}$$

and observe that

$$\mathrm{E}\left[\, y_h(t+s)\, e_i(t) \,\right] = 0 \quad \text{for } i \neq h; \tag{4.11.3}$$

because of the assumed independence between $e_i(t)$ and $e_j(t)$ for $i \neq j$, and that

$$\mathrm{E}\left[\, y_i(t+s)\, e_i(t) \,\right] = 0 \quad \text{for } s > 0 \tag{4.11.4}$$

because of the whiteness of $e_i(t)$. Define now the quantities

$$r_s^{ij} = \mathrm{E}\left[\, y_i(t+s)\, y_j(t) \,\right]; \tag{4.11.5}$$

equation (4.11.2) can be rewritten in the form

$$\begin{aligned} r^{hi}_{s-\nu_i} - \alpha_{i11}\, r^{h1}_{s} - \ldots - \alpha_{i1\nu_{i1}}\, r^{h1}_{s-\nu_{i1}+1} - \ldots - \alpha_{ii1}\, r^{hi}_{s} - \ldots \\ - \alpha_{ii\nu_i}\, r^{hi}_{s-\nu_i+1} - \ldots - \alpha_{im1}\, r^{hm}_{s} - \ldots - \alpha_{im\nu_{im}}\, r^{hm}_{s-\nu_{im}+1} = 0. \end{aligned} \tag{4.11.6}$$

Substituting in equation (4.11.2) $y_h(t+s)$ with $y_1(t)\dots y_1(t+\nu_{i1}-1)\dots y_m(t)\dots y_m(t+\nu_{im}-1)$, we obtain a set of $\ell=\sum_{j=1}^m \nu_{ij}$ equations that can be written in the form

$$R_i\,\theta_i^\circ = \rho_i \tag{4.11.7}$$

where θ_i° is defined by (4.10.2) and R_i and ρ_i are given by

$$R_i = \begin{bmatrix} r_0^{11} & r_1^{11} & \dots & r_{\nu_{i1}-1}^{11} & \dots & r_0^{1i} & \dots & r_{-\nu_i+1}^{1i} & \dots & r_0^{1m} & \dots & r_{-\nu_{im}+1}^{1m} \\ r_1^{11} & r_0^{11} & \dots & r_{\nu_{i1}-2}^{11} & \dots & r_1^{1i} & \dots & r_{-\nu_i+2}^{1i} & \dots & r_1^{1m} & \dots & r_{-\nu_{im}+2}^{1m} \\ \vdots & & & & & & & & & & & \vdots \\ r_0^{i1} & r_{-1}^{i1} & \dots & r_{-\nu_{i1}+1}^{i1} & \dots & r_0^{ii} & \dots & r_{\nu_i-1}^{ii} & \dots & r_0^{im} & \dots & r_{-\nu_{im}+1}^{im} \\ \vdots & & & & & & & & & & & \vdots \\ r_{\nu_{im}-1}^{m1} & r_{\nu_{im}-2}^{m1} & \dots & r_{\nu_{im}-\nu_{i1}}^{m1} & \dots & r_{\nu_{im}-1}^{mi} & \dots & r_{\nu_{im}-\nu_i}^{mi} & \dots & r_{\nu_{im}-1}^{mm} & \dots & r_0^{mm} \end{bmatrix} \tag{4.11.8}$$

$$\rho_i = \left[\, r_{\nu_i}^{i1} \quad r_{\nu_i-1}^{i1} \quad \dots \quad r_{\nu_i-\nu_{i1}+1}^{i1} \quad \dots \quad r_{\nu_i}^{ii} \quad \dots \quad r_1^{ii} \quad \dots \quad r_{\nu_i}^{im} \quad \dots \quad r_{\nu_i-\nu_{im}+1}^{im} \,\right]^T . \tag{4.11.9}$$

Equation (4.11.7) defines the minimal Yule–Walker estimate

$$\theta_i^\circ = R_i^{-1}\,\rho_i \tag{4.11.10}$$

based on the same time shifts as least squares estimates; larger time shifts and/or overdetermined Yule–Walker estimates can be easily derived from (4.11.6).

Remark 4.11.1 – It can be noted that only the blocks on the main diagonal of R_i have a Toeplitz structure because, in general,

$$\mathrm{E}\left[\, y_i(t+s)\, y_j(t) \,\right] \neq \mathrm{E}\left[\, y_i(t)\, y_j(t+s) \,\right] \quad \text{for } i \neq j\,. \tag{4.11.11}$$