ST

System Theory

ST.6 REALIZATION OF MIMO INPUT–OUTPUT SEQUENCES

The extension of Problem ST.5.1 to the MIMO case is straightforw ard Its solution is based on the structural and algebraic properties of canonical input–output models (ST.4.17).

Problem ST.6.1 (Minimal realization of MIMO systems) – Given generic input–output sequences of a linear discrete–time multi–output dynamic system, determine aminimal state space model compatible with the sequences and its initial state.

Solution – Define the composite matrix

$$H^L(k_1,\dots,k_m) = \left[\, H^L_{k_1}(y_1)\ \dots\ H^L_{k_m}(y_m)\ H^L_{k_M}(u_1)\ \dots\ H^L_{k_M}(u_r) \,\right] \qquad \text{(ST.6.1)}$$

where $k_M = \max_i\,(k_i)$, $(i = 1,\dots,m)$, and test the rank of the matrices

$$H^L(2,1,\dots,1)\ H^L(2,2,\dots,1)\ \dots\ H^L(2,2,\dots,2)\ H^L(3,2,\dots,2)\ \dots \qquad \text{(ST.6.2)}$$

where L is selected as in Problem ST.5.1. As soon as a deficient–rank matrix is found, one of the m relations (ST.4.17) is defined. If the first matrix exhibiting a rank deficiency is

$$H^L(k,\dots,\underset{\substack{\uparrow\\ i}}{k},k-1,\dots,k-1)$$

then $\nu_i = k-1$. The dependence coefficients of the last vector of $H^L_k(y_i)$ from the columns of the submatrices $H^L_*(y_j)$ are the parameters α_{ijk} while the dependence coefficients from the columns of the matrices $H^L_*(u_j)$ are the parameters β_{ijk}. The i–th argument of $H^L(k_1,\dots k_m)$ is then set at ν_i and no longer increased while remaining ones are increased according to the scheme (ST.6.2) until a second, $\dots$, m–th rank deficient matrix is detected.

The procedure determines the integers ν_i and the scalars α_{ijk} and β_{ijk} that define the polynomial matrices $Q(z)$ and $P(z)$. The matrices $\tilde{A}$ and $\tilde{C}$ can then be directly written, as well as $\overline{B}$ (ST.4.13) and M (ST.4.16) which lead, by inverting M, to $\Phi = M^{-1}\overline{B}$ whose entries define $\tilde{B}$ and $\tilde{D}$. The initial state can then be computed using relation (ST.4.2).

Remark ST.6.1 – In the realization of purely dynamic systems, the integer k_M appearing in the definition of $H^L(k_1, \ldots, k_m)$ (ST.6.1) will be defined as $k_M = \max_i (k_i) - 1$, $(i = 1, \ldots, m)$.

Example – A sequence of 20 input–output samples generated by a linear discrete–time dynamic system is given by

$$u_1(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad u_2(t) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad y_1(t) = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2.7380952 \\ 0.17460317 \\ 0.060185185 \\ 3.0147707 \\ -0.64812978 \\ 0.83313737 \\ -2.1928732 \\ 1.1973739 \\ 0.49884006 \\ 1.3851584 \\ 2.9116398 \\ 0.22884872 \\ 0.065360969 \\ 3.0051526 \\ -0.65923195 \\ 0.82444990 \\ -2.1987572 \end{bmatrix} \quad y_2(t) = \begin{bmatrix} 3 \\ 2 \\ 0.071428571 \\ 2.4166667 \\ 1.7876984 \\ 4.0710979 \\ -0.85510362 \\ -3.9299034 \\ -1.5342247 \\ 2.9091892 \\ 2.3576037 \\ 2.3723300 \\ 0.38151607 \\ 2.5772667 \\ 1.8579335 \\ 4.0988881 \\ -0.84497508 \\ -3.9265360 \\ -1.5332537 \\ 2.9093880 \end{bmatrix} .$$

A minimal state space model compatible with these data and its initial state will be now computed. As a first step we will construct the sequence of matrices (ST.6.2) and test their rank. To take into account the inevitable approximations of the values of the available samples and to rely on numerically robust procedures we will compute their condition number (given by the ratio between the largest and the smaller singular values). Taking $L = 17$ we obtain:

$$H^{17}(2,1) \to 19.85$$
$$H^{17}(2,2) \to 434.5$$
$$H^{17}(3,2) \to 1102$$
$$H^{17}(3,3) \to 4.930E+5$$

The condition number of $H^{17}(3,3)$ shows that this matrix must be considered as rank deficient and this indicates that $\nu_2 = 2$. The subsequent matrix to be tested is

$$H^{17}(4,2) \rightarrow 4.393E+5;$$

this value indicates that also $H^{17}(4,2)$ must be considered as rank deficient so that $\nu_1 = 3$. The dependence coefficients between the columns of $H^{17}(3,3)$ and $H^{17}(4,2)$ can be determined in many different ways; computing, for instance, a basis of the kernels of $H^{17}(3,3)$ and $H^{17}(4,2)$ and normalizing to -1 their sixth and fourth entries we find the following vectors of coefficients

$$\begin{bmatrix} \alpha_{211} \\ \alpha_{212} \\ \alpha_{213} \\ \alpha_{221} \\ \alpha_{222} \\ -1 \\ \beta_{211} \\ \beta_{212} \\ \beta_{213} \\ \beta_{221} \\ \beta_{222} \\ \beta_{223} \end{bmatrix} = \begin{bmatrix} 0.0238 \\ -0.1190 \\ 0.1429 \\ -0.1190 \\ 0.6905 \\ -1 \\ 0.0238 \\ 1.7381 \\ 1.8571 \\ -0.4762 \\ 1.6905 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} \alpha_{111} \\ \alpha_{112} \\ \alpha_{113} \\ -1 \\ \alpha_{121} \\ \alpha_{122} \\ \beta_{111} \\ \beta_{112} \\ \beta_{113} \\ \beta_{114} \\ \beta_{121} \\ \beta_{122} \\ \beta_{123} \\ \beta_{124} \end{bmatrix} = \begin{bmatrix} 0.0794 \\ -0.5635 \\ 1.3095 \\ -1 \\ -0.0079 \\ 0.0238 \\ -1.0873 \\ 2.5159 \\ -1.3095 \\ 1.0000 \\ -0.0317 \\ 0.0238 \\ 0 \\ 0 \end{bmatrix}.$$

It is now possible to write the input–output model

$$Q(z) = \begin{bmatrix} z^3 - 1.3095\,z^2 + 0.5635\,z - 0.0794 & -0.0238\,z + 0.0079 \\ -0.1429\,z^2 + 0.1190\,z - 0.0238 & z^2 - 0.6905\,z + 0.1190 \end{bmatrix}$$

$$P(z) = \begin{bmatrix} z^3 - 1.3095\,z^2 + 2.5159\,z - 0.0794 & 0.0238\,z - 0.0317 \\ 1.8571\,z^2 + 1.7381\,z + 0.0238 & -z^2 + 1.6905\,z - 0.4762 \end{bmatrix}$$

as well as the matrices

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.0794 & -0.5635 & 1.3095 & -0.0079 & 0.0238 \\ 0 & 0 & 0 & 0 & 1 \\ 0.0238 & -0.1190 & 0.1429 & -0.1190 & 0.6905 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -0.0794 & 0.5635 & -1.3095 & 1 & 0.0079 & -0.0238 & 0 \\ 0.5635 & -1.3095 & 1 & 0 & -0.0238 & 0 & 0 \\ -1.3095 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0238 & 0.1190 & -0.1429 & 0 & 0.1190 & -0.6905 & 1 \\ 0.1190 & -0.1429 & 0 & 0 & -0.6905 & 1 & 0 \\ -0.1429 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} -1.0873 & -0.0317 \\ 2.5159 & 0.0238 \\ -1.3095 & 0 \\ 1 & 0 \\ 0.0238 & -0.4762 \\ 1.7381 & 1.6905 \\ 1.8571 & -1 \end{bmatrix}.$$

Inverting M we finally obtain

$$\Phi = M^{-1}\overline{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 0 \\ 1.6667 & 0 \\ 2 & -1 \\ 3 & 1 \\ 2.1667 & 0.3333 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 1.6667 & 0 \\ 3 & 1 \\ 2.1667 & 0.3333 \end{bmatrix} \quad \tilde{D} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}.$$

The corresponding initial state is given by

$$x_0 = \begin{bmatrix} 1 & 0 \\ z & 0 \\ z^2 & 0 \\ 0 & 1 \\ 0 & z \end{bmatrix} y(1) - \begin{bmatrix} d_1 & 0 & 0 \\ b_{11} & d_1 & 0 \\ b_{12} & b_{11} & d_1 \\ d_2 & 0 & 0 \\ b_{21} & d_2 & 0 \end{bmatrix} \begin{bmatrix} I \\ zI \\ z^2 I \end{bmatrix} u(1)$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 2 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$