ID7

ARMA Identification

7.4 EXAMPLE 7.1

The sequence considered in this example is given by 520 samples generated by the ARMA process

$$\begin{aligned} y(t) = {} & 2\,y(t-1) - 1.68\,y(t-2) + 0.576\,y(t-3) \\ & + w(t) - 1.4\,w(t-1) + 0.59\,w(t-2) - 0.07\,w(t-3). \end{aligned} \tag{7.4.1}$$

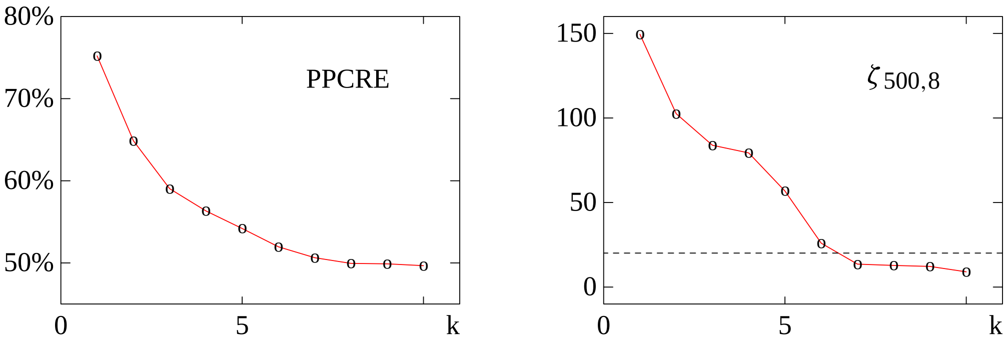
$w(t)$ is a Gaussian and white process with zero mean and variance, computed on the first 500 samples, $\sigma_w^2 = 0.0265$. The variance of the output sequence is $\sigma_y^2 = 0.0435$, while the variance of the equation error is $\sigma_e^2 = 0.0902$. The autoregressive parameters α_i will be now computed using Yule–Walker equations for a model with the same order of the process, $n = 3$. We can note that Yule–Walker equations (7.3.6) written for $k = 4, 5$ and 6 give the same estimate as the IV algorithm (6.3.6) taking

$$Z = \begin{bmatrix} y(1) & y(2) & y(3) \\ \vdots & \vdots & \vdots \\ y(N) & y(N+1) & y(N+2) \end{bmatrix}, \quad H = \begin{bmatrix} y(4) & y(5) & y(6) \\ \vdots & \vdots & \vdots \\ y(N+3) & y(N+4) & y(N+5) \end{bmatrix}$$

and $y^\circ = [y(7), \ldots, y(N+6)]^T$. For $N = 500$ we obtain the estimate

$$\begin{aligned} \alpha_1 &= 0.6679\ (0.576) \\ \alpha_2 &= -1.8883\ (-1.68) \\ \alpha_3 &= 2.1943\ (2). \end{aligned}$$

These estimates allow computing, by means of (7.3.8), the sequence of equation errors to be modeled as a MA process. The first test to be performed on this sequence concerns the order of an auxiliary AR model.



Figures 7.4.1 and 7.4.2 – PPCRE and $\zeta_{500,8}$ for AR models with order 1–10

The PPCRE test reported in Figure 7.4.1 for AR models with orders between 1 and 10 suggests orders not lower than 7. This indication is confirmed by the χ^2 whiteness test on the residuals reported in Figure 7.4.2 where it can be observed that $\zeta_{500,8}$ is lower than the 99% confidence level for $\chi^2(8)$ (dashed line) only for models with orders larger than 6. Selecting 7 as order of the auxiliary AR model we obtain, using least squares, the AR model

$$\begin{aligned} e(t) = & -1.5492\, e(t-1) - 1.7413\, e(t-2) - 1.6441\, e(t-3) - 1.3706\, e(t-4) \\ & -1.0126\, e(t-5) - 0.6037\, e(t-6) - 0.2193\, e(t-7) + w(t) \end{aligned}$$

whose residuals approximate the remote white sequence $w(t)$. The variance of the obtained sequence is

$$\hat{\sigma}_w^2 = 0.0283,$$

which constitutes a good approximation of the true value (0.0265). It is now possible to estimate the parameters of the MA part of the model; using least squares we obtain

$$\begin{aligned} \gamma_1 &= 0.0344\ (-0.07) \\ \gamma_2 &= 0.6610\ (0.59) \\ \gamma_3 &= -1.5228\ (-1.4). \end{aligned}$$

The output prevision given by predictor (7.2.1) is plotted in Figure 7.4.3 (black line) against observed values. The residuals are reported in Figure 7.4.4.

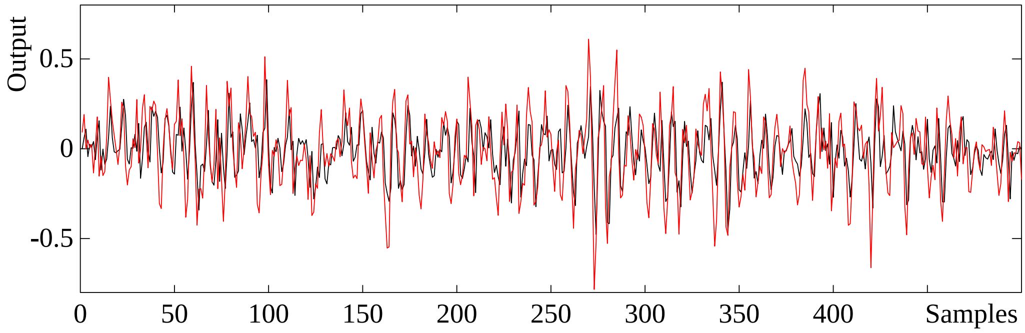


Figure 7.4.3 – Model prevision (black line) and observed output

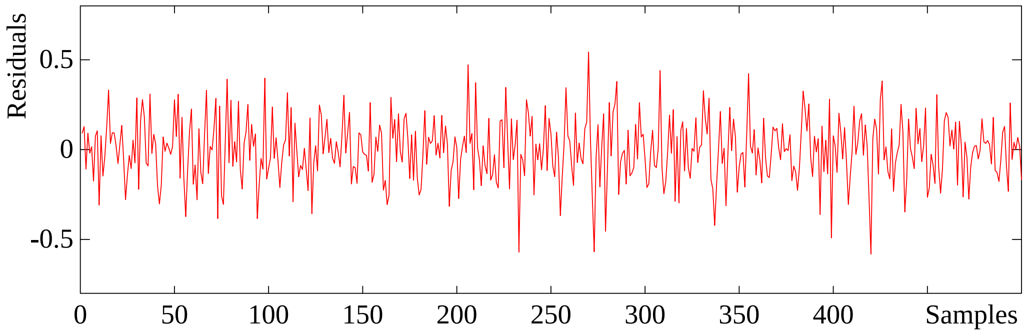


Figure 7.4.4 – Residuals of the ARMA model

The variance of the prediction error is

$$\sigma_\varepsilon^2 = 0.0296.$$

Performing a whiteness test on the residuals to validate the model, we obtain $\zeta_{500,8} = 19.0138$, i.e. a value not far from the 99% confidence level for $\chi^2(8)$. We will try now to improve the model applying the Gauss–Newton algorithm in order to obtain a PEM estimate of the model. By using the constant value $\Delta_k = 1/\sqrt{2}$ in (6.13.19), after 11 iterations we obtain the estimate

$$\begin{aligned}
\alpha_1 &= 0.6588\ (0.576) & \gamma_1 &= -0.0661\ (-0.07)\\
\alpha_2 &= -1.7702\ (-1.68) & \gamma_2 &= 0.6390\ (0.59)\\
\alpha_3 &= 2.1113\ (2) & \gamma_3 &= -1.5690\ (-1.4).
\end{aligned}$$

The variance of the corresponding prevision error is

$$\sigma_\varepsilon^2 = 0.0260,$$

almost equal to the value of σ_w^2. The covariance matrix (6.15.4) of this estimate is

$$\operatorname{cov}\theta_{500}^{\circ} = 10^{-2}\begin{bmatrix} 0.270 & -0.427 & 0.159 & -0.271 & 0.439 & -0.169 \\ -0.427 & 0.902 & -0.477 & 0.337 & -0.832 & 0.496 \\ 0.159 & -0.477 & 0.319 & -0.068 & 0.395 & -0.328 \\ -0.271 & 0.337 & -0.068 & 0.509 & -0.690 & 0.184 \\ 0.439 & -0.832 & 0.395 & -0.690 & 1.406 & -0.718 \\ -0.169 & 0.496 & -0.328 & 0.184 & -0.718 & 0.536 \end{bmatrix}$$

and the corresponding standard deviations of the parameters are

$$\begin{aligned} \operatorname{std}\alpha_1 &= 0.052\ (0.0828) & \qquad \operatorname{std}\gamma_1 &= 0.071\ (0.0039) \\ \operatorname{std}\alpha_2 &= 0.095\ (0.0902) & \qquad \operatorname{std}\gamma_2 &= 0.119\ (0.0490) \\ \operatorname{std}\alpha_3 &= 0.057\ (0.1113) & \qquad \operatorname{std}\gamma_3 &= 0.073\ (0.1690). \end{aligned}$$

The output prediction of this PEM model is reported in Figure 7.4.5 (black line) against observed values. The residuals are plotted in Figure 7.4.6 and the whiteness test gives, finally, $\zeta_{500,8} = 1.9102$, confirming the excellent behavior of the model.

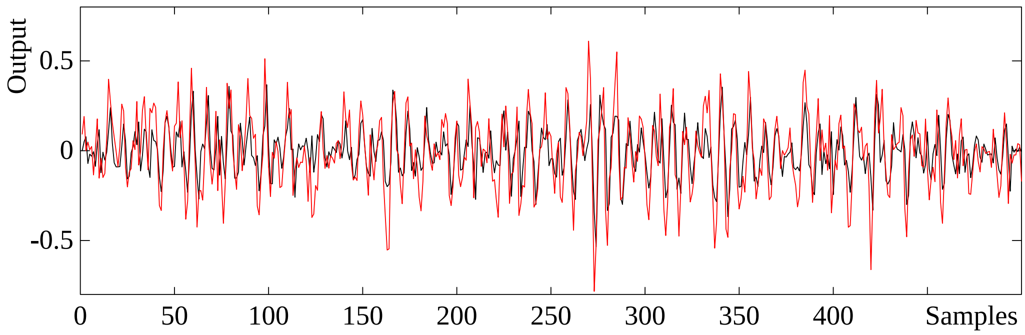


Figure 7.4.5 – PEM model prevision (black line) and observed output

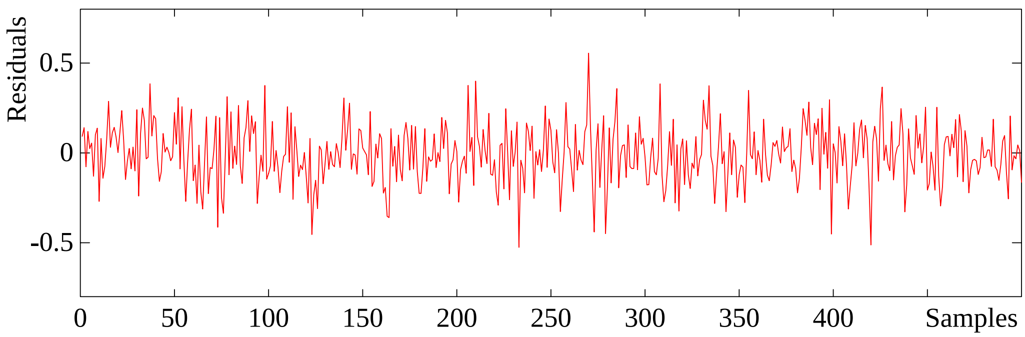


Figure 7.4.6 – Residuals of the PEM ARMA model