ID3

ARX Identification

3.15 EXAMPLE 3.1

The ARX process considered in this example is described by the model

$$y(t) = 0.6\, y(t-1) - 0.34\, y(t-2) + 0.4564\, u(t-1) + 0.2738\, u(t-2) + e(t). \quad (3.15.1)$$

The input sequence, reported in Figure 3.15.1, has null mean value and variance $\sigma_u^2 = 1$.

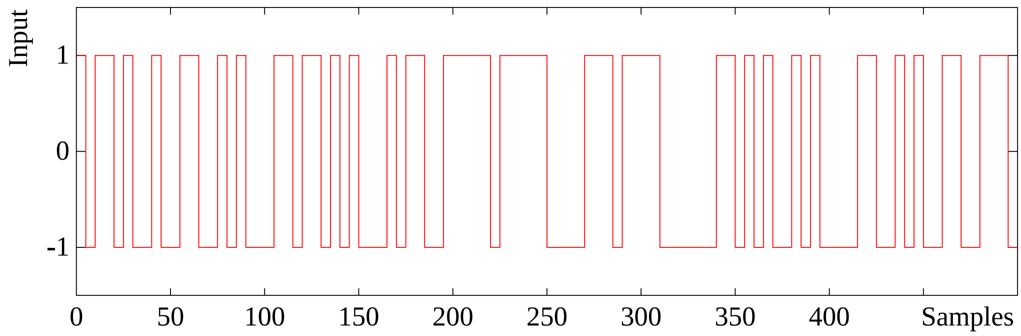


Figure 3.15.1 – Input sequence of process (3.15.1)

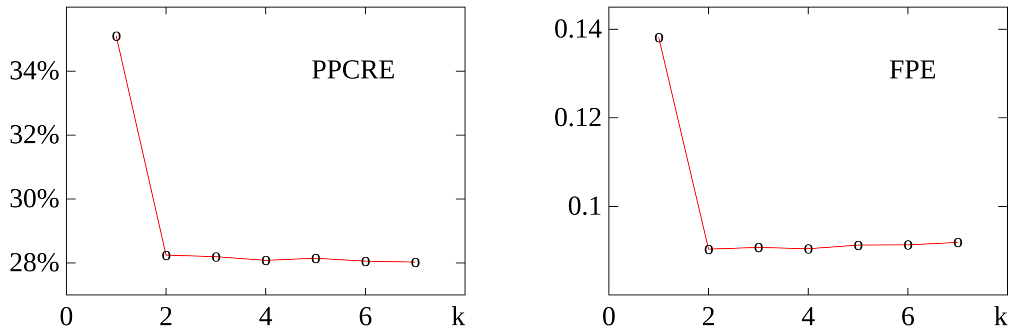
The variance of the output sequence is, in absence of noise ($e(t) = 0$), $\sigma_{y*}^2 = 1$; $e(\cdot)$ is a stationary and Gaussian process with null expected value and variance $\sigma_e^2 = 0.09$. With reference to the decomposition of Figure 3.1.2, the colored noise $v(t)$ has a variance, computed as previous ones on the whole set of 510 samples, given by $\sigma_v^2 = 0.125$; these values can be interpreted as the presence, on the data, of an amount of noise of

$$100\, \frac{\sigma_v}{\sigma_{y*}} = 35.4\,\%.$$

Since our data have been generated by simulation, it will be possible to compare the results given by identification with the real description of the process.

3.15.1 Determination of the model order

All order determination criteria described in the following have been applied assuming $N = 500$. The values of the PPCRE (3.14.14) computed for $k = 1, \ldots, 7$ are plotted in Figure 3.15.2. It is possible to observe a sensible decrease in the PPCRE passing from $k = 1$ to $k = 2$ and a subsequent stabilization. This criterion leads thus to select $n = 2$ as most suitable choice for the model order.



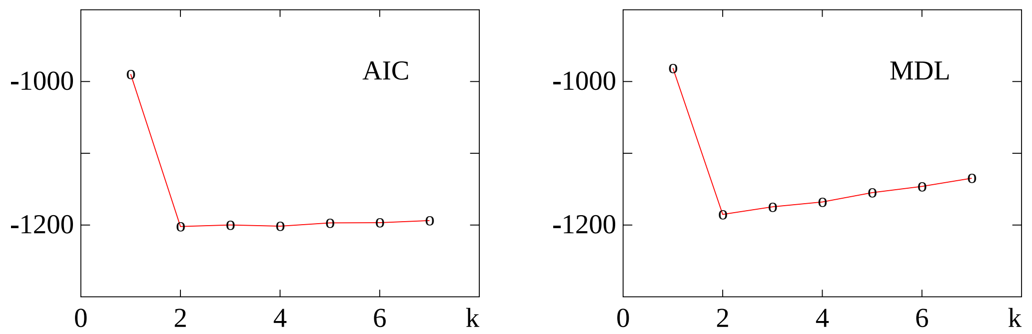
Figures 3.15.2 and 3.15.3 – PPCRE and FPE criteria for $N = 500$

Estimating the variance of $e(\cdot)$ by means of (3.14.16) for $k = 2$ we obtain

$$\hat{\sigma}_e^2 = 0.0896,$$

i.e. a value that approximates very well the true one (0.09).

The results of the FPE criterion are reported in Figure 3.15.3. The minimum of this criterion occurs again for $k = 2$; another local minimum occurs for $k = 4$.



Figures 3.15.4 and 3.15.5 – AIC and MDL criteria for $N = 500$

A similar behavior can be observed on AIC (Figure 3.15.4). A clear indication is finally given by MDL (Figure 3.15.5) which indicates $k = 2$ as the only model order compatible with the data.

All criteria allow thus, in this case, to deduce correctly the order of the model; in correspondence with this order we obtain also a very accurate estimate of the variance of $e(t)$.

3.15.2 Parameter estimate

A first estimate has been performed by means of the least squares algorithm, (3.3.12) for $N = 50$, i.e. using only one tenth of the available data. The parameter values obtained are

$$\begin{aligned} \alpha_1 &= -0.3197\ (-0.34) \\ \alpha_2 &= \ \ \ 0.5073\ (0.6) \\ \beta_1 &= \ \ \ 0.3369\ (0.2738) \\ \beta_2 &= \ \ \ 0.4800\ (0.4564). \end{aligned}$$

The mean square prevision error (3.3.2) of this model is

$$J(\theta^\circ_{50}) = 0.0894$$

and the corresponding estimate of the variance of $e(\cdot)$, given by (3.10.5) is $\hat{\sigma}_e^2 = 0.097$. The observed output is compared with the one–step–ahead prevision of the model (black line) on the first 50 samples in Figure 3.15.6. The corresponding residuals are plotted in Figure 3.15.7.

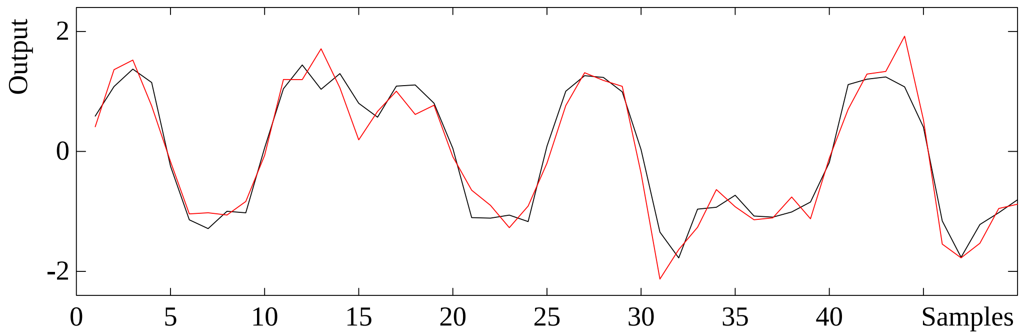


Figure 3.15.6 – Model prevision (black line) and observed output

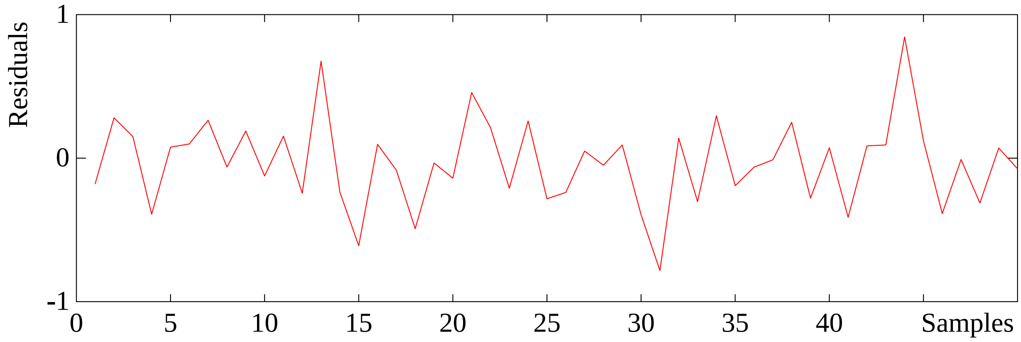


Figure 3.15.7 – Residuals of the model identified from the first 50 samples

Estimating the parameters on the whole set of data ($N = 500$) we obtain the following values

$$\begin{aligned} \alpha_1 &= -0.3203\ (-0.34) \\ \alpha_2 &= \ \ \ 0.6016\ (0.6) \\ \beta_1 &= \ \ \ 0.2499\ (0.2738) \\ \beta_2 &= \ \ \ 0.4576\ (0.4564). \end{aligned}$$

The covariance matrix (3.8.5) of the estimate, computed using the estimated value $\hat{\sigma}_e^2 = 0.0896$, is

$$\operatorname{cov}\theta^\circ_{500} = 10^{-3}\begin{bmatrix} 0.568 & -0.592 & 0.080 & 0.082 \\ -0.592 & 1.313 & -0.755 & -0.050 \\ 0.080 & -0.755 & 1.131 & -0.395 \\ 0.082 & -0.050 & -0.395 & 0.486 \end{bmatrix}.$$

The standard deviations asociated with the estimates of single parameters are thus given by

$$\begin{aligned} &\operatorname{std}\alpha_1 = 0.024\ (0.0197) && \operatorname{std}\beta_1 = 0.034\ (0.0239) \\ &\operatorname{std}\alpha_2 = 0.036\ (0.0016) && \operatorname{std}\beta_2 = 0.022\ (0.0012). \end{aligned}$$

It can be noted that actual deviations (reported in parentheses), show very good agreements with these values and with the assumption of Gaussian distribution for the estimates. The mean square prevision error associated with this model is

$$J(\theta^\circ_{500}) = 0.0889.$$

Figure 3.15.8 reports the one–step–ahead prevision of the model (black line) against observed values; the residuals are plotted in Figure 3.15.9.

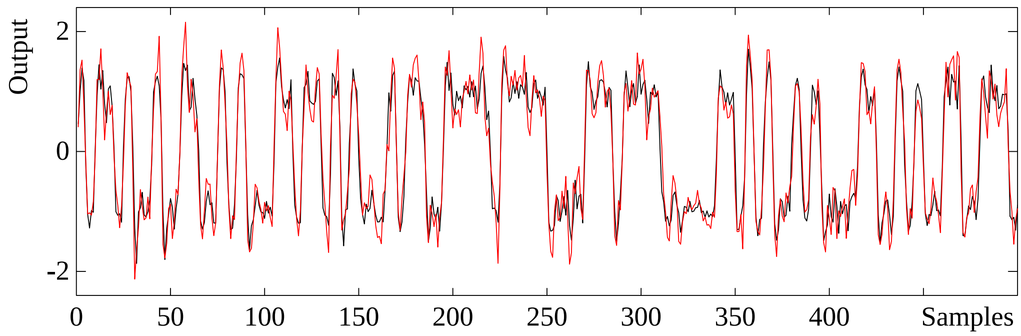


Figure 3.15.8 – Model prevision (black line) and observed output

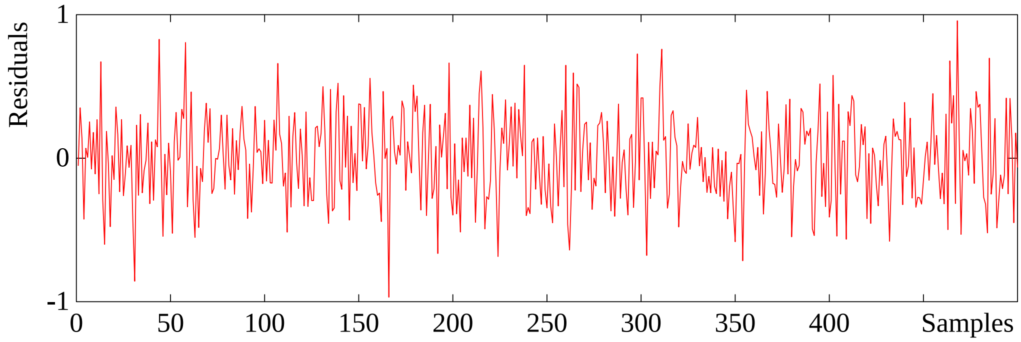


Figure 3.15.9 – Residuals of the model identified from the whole set of data

This model has then been used to perform a complete simulation, using the first two observed samples of the output and the complete input sequence; the obtained values are plotted in Figure 3.15.10 (black line) against observed ones. With reference to the decomposition of an ARX process into deterministic and stochastic processes (Figure 3.1.2), the obtained values should approximate the output $y(t)^*$, of the deterministic part of the process. The difference between $y(t)^*$ and the observed output sequence $y(t)$ is a reconstruction of the sequence of colored noise $v(t)$; comparing its plot, reported in Figure 3.15.11, with the plot of $e(t)$ which is essentially white (Figure 3.15.9) it is possible to appreciate the presence of some correlation between the samples.

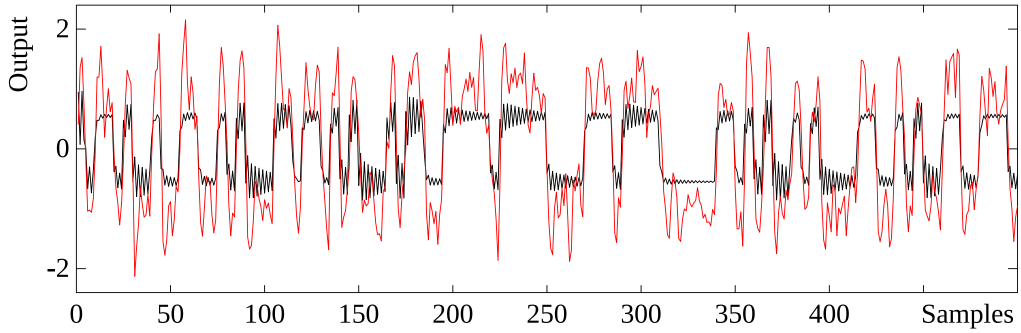


Figure 3.15.10 – Complete simulation (black line) and observed output

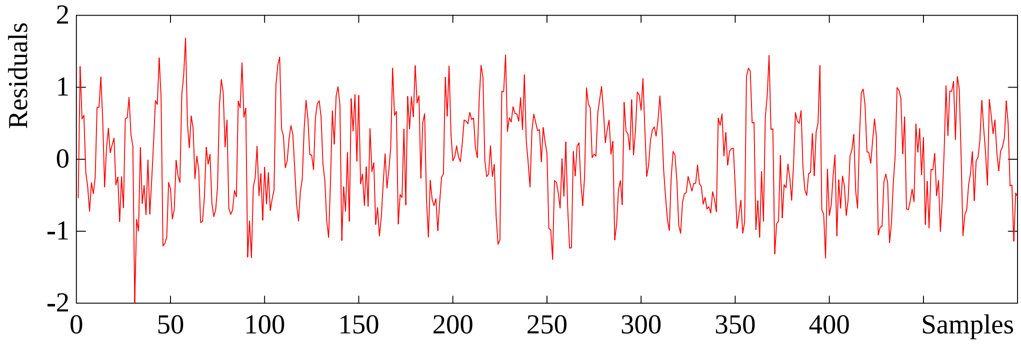


Figure 3.15.11 – Residuals of the complete simulation

3.15.3 Model validation

A first validation has been performed testing the residual whiteness; computing the sample covariances $R_\varepsilon^{500}(\tau)$ (3.14.22) for $\tau = 0, \dots, 8$, we obtain the following value for $\zeta_{500,8}$

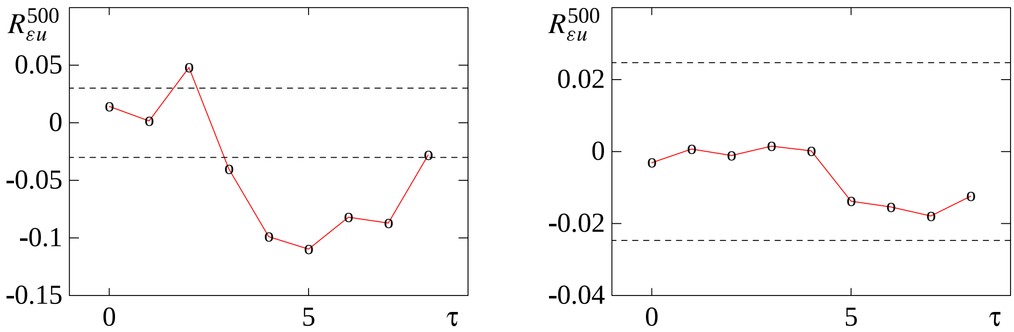
$$\zeta_{500,8} = 7.51.$$

Adopting a confidence level of 99%, the corresponding level of χ^2_α for $M = 8$ is

$$\chi^2_\alpha(8) = 20.1;$$

since $\zeta_{500,8} < \chi^2_\alpha(8)$ it is possible to establish that the computed value fits very well the assumption of whiteness for the residuals. Performing the same test on the residuals of an order 1 model we would find $\zeta_{500,8} = 54.7 > \chi^2_\alpha(8)$; this would lead to rejecting the assumption of whiteness and the choice of the corresponding model.

A second validation has been performed by computing the correlations $R^{500}_{\varepsilon u}(\tau)$ (3.14.24) between the residuals and the input sequence. Figures 3.15.12 and 3.15.13 show the corresponding diagrams for models with order 1 and 2 and for $\tau = 0, \dots, 8$. The horizontal lines on the plots correspond to confidency levels of 95% for a Gaussian distribution; $\sigma^2_{\varepsilon u}$ has been computed using (3.14.25) with $k = -8, \dots, 8$.



Figures 3.15.12 and 3.15.13 – Correlation between input and residuals for order 1 and 2 models

It can be observed that the values of $R^{500}_{\varepsilon u}(\tau)$ obtained for an order 1 model go well beyond the considered confidence interval while those corresponding to an order 2 model remain always inside. This test confirms that an order 1 model is unsuitable to describe the considered process while an order 2 model can be successfully validated.

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