ID4

AR Identification

4.6 LEVINSON ALGORITHM

It has alreday been observed that it is useful, in some applications, to compute families of models with increasing order for the same AR process taking advantage of all previous computations. Considering the minimal Yule–Walker estimate it is possible to take advantage of the Toeplitz structure of R_n to deduce a simple expression linking the parameters of a model to those of previous one.

Assume that the parameters, θ_R° of an order n model and the estimate (4.5.12) of σ_e^2 have already been computed; these quantities will be denoted in the following with θ_{Rn}° and σ_{en}^2 to avoid any confusion with those, $\theta_{R(n+1)}^\circ$ and $\sigma_{e(n+1)}^2$ associated with a model with order $n+1$. For the same reason we will introduce the notations α_i^n and $\alpha_i^{(n+1)}$ for the entries of θ_{Rn}° and $\theta_{R(n+1)}^\circ$. The parameter vector $\theta_{R(n+1)}^\circ$ is given by

$$\theta_{R(n+1)}^\circ = R_{n+1}^{-1}\,\rho_{n+1} = \begin{bmatrix} R_n & \rho_{Rn} \\ \rho_{Rn}^T & r_0 \end{bmatrix}^{-1} \begin{bmatrix} \rho_n \\ r_{n+1} \end{bmatrix} \tag{4.6.1}$$

where $\rho_{Rn} = [r_n \ \ldots \ r_1]^T$ denotes a vector having the same entries of ρ_n but in reverse order; because of the Toeplitz structure of R_n, relation (4.5.9) implies that

$$\theta_n^\circ = R_n^{-1}\rho_{Rn}. \tag{4.6.2}$$

Applying to R_{n+1} a well known lemma on the inversion of partitioned matrices we obtain

$$R_{n+1}^{-1} = \begin{bmatrix} R_n^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -R_n^{-1}\,\rho_{Rn} \\ 1 \end{bmatrix} \left[r_0 - \rho_{Rn}^T R_n^{-1}\,\rho_{Rn} \right]^{-1} \left[-\rho_{Rn}^T\,R_n^{-1} \quad 1 \right]$$

$$= \frac{\begin{bmatrix} \sigma_{en}^2 R_n^{-1} + \theta_n^\circ \theta_n^{\circ T} & -\theta_n^\circ \\ -\theta_n^{\circ T} & 1 \end{bmatrix}}{\sigma_{en}^2}; \tag{4.6.3}$$

expression (4.6.3), quite similar to (4.4.5), has been obtained expanding the denominator expression

$$r_0 - \rho_{Rn}^T R_n^{-1} \rho_{Rn} = r_0 - \theta_n^{\circ T} \rho_{Rn} = r_0 - \alpha_n^n r_1 - \ldots - \alpha_1^n r_n = \sigma_{en}^2. \tag{4.6.4}$$

Substituting the expression of R_{n+1}^{-1} in (4.6.1) we obtain, with simple passages, the following expressions

$$\alpha_1^{n+1} = \frac{1}{\sigma_{en}^2} \left[r_{n+1} - \sum_{j=1}^{n} \alpha_j^n r_j \right] \tag{4.6.5a}$$

$$\alpha_i^{n+1} = \alpha_{i-1}^n - \alpha_{n-i+2}^n \alpha_1^{n+1} \qquad (i = 2, \ldots, n+1). \tag{4.6.5b}$$

$\sigma_{e(n+1)}^2$ can be finally obtained by direct substitution of (4.6.5b) in (4.5.12), written for $n+1$; we find

$$\begin{aligned} \sigma_{e(n+1)}^2 &= r_0 - \alpha_{n+1}^{n+1} r_1 - \ldots - \alpha_2^{n+1} r_n - \alpha_1^{n+1} r_{n+1} \\ &= r_0 - \left(\alpha_n^n - \alpha_1^{n+1} \alpha_1^n\right) r_1 - \ldots - \left(\alpha_1^n - \alpha_1^{n+1} \alpha_n^n\right) r_n - \alpha_1^{n+1} r_{n+1} \\ &= \sigma_{en}^2 + \alpha_1^{n+1} \left[\sum_{i=1}^{n} \alpha_i^n r_i - r_{n+1} \right] = \sigma_{en}^2 \left[1 - \left(\alpha_1^{n+1}\right)^2 \right]. \end{aligned} \tag{4.6.6}$$

Note that relation (4.6.6) shows how the equation error $e(t)$ decreases increasing the order of the model. Note also that previous relations, which constitute Levinson algorithm, can be used only if the matrices R_k have a Toeplitz structure. This condition is not satisfied using the sample matrix R_m^s (4.5.13) because, as already noted, its entries are computed from finite shifted samples; when N is large this aspect does play only a secondary role and it is possible to force a Toeplitz structure on R_m^s computing sample estimates of its entries from the whole sequence.