ID7

ARMA Identification

7.3 ESTIMATION OF ARMA MODELS

ARMA models can be seen as ARMAX models without inputs; it is thus possible to use, to estimate the α_i and γ_i parameters, IV and PE methods but not to use past inputs as instruments or to construct instruments with models of type (6.3.8). It is instead possible to estimate the autoregressive part of the model by means of Yule–Walker equations. Rewrite, for this purpose model (7.1.1) in the form

$$y(t)-\alpha_n y(t-1)-\ldots-\alpha_1 y(t-n)=w(t)+\gamma_n w(t-1)+\ldots+\gamma_1 w(t-n) \quad (7.3.1)$$

and consider the expected values of both members of (7.3.1) multiplied by $y(t-k)$; we obtain

$$\begin{aligned} &\mathrm{E}\left[\, y(t)\, y(t-k)-\alpha_n\, y(t-1)\, y(t-k)-\ \ldots\ -\alpha_1\, y(t-n)\, y(t-k)\,\right]= \\ &\mathrm{E}\left[\, w(t)\, y(t-k)+\gamma_n\, w(t-1)\, y(t-k)+\ \ldots\ +\gamma_1\, w(t-n)\, y(t-k)\,\right]. \end{aligned} \quad (7.3.2)$$

Note now that

$$\mathrm{E}\left[\, w(t-i\,)\, y(t-k)\,\right]=0 \ \text{ for } \ k>i, \quad (7.3.3)$$

and observe that on the right side of (7.3.2), the most delayed noise term is $w(t-n)$; taking $k>n$ we thus obtain

$$\mathrm{E}\left[\, y(t)\, y(t-k)-\alpha_n\, y(t-1)\, y(t-k)-\ \ldots\ -\alpha_1\, y(t-n)\, y(t-k)\,\right]=0. \quad (7.3.4)$$

Defining the covariances

$$r_k=\mathrm{E}\left[\, y(t)\, y(t-k)\,\right], \quad (7.3.5)$$

relation (7.3.4) assumes the form

$$r_k-\alpha_n\, r_{k-1}-\alpha_{n-1}\, r_{k-2}-\ldots-\alpha_1\, r_{k-n}=0 \quad \text{for} \ \ k=n+1, n+2, \ldots \quad (7.3.6)$$

Yule–Walker equations (7.3.6) are analogous to (4.5.5) introduced for AR models and can be used to estimate the autoregressive parameters α_i using expressions (4.5.9) or (4.5.10) corresponding to minimal and overdetermined estimates; R_m and ρ_m are now given by

$$R_m = \begin{bmatrix} r_n & r_{n-1} & \dots & r_1 \\ r_{n+1} & r_n & \dots & r_2 \\ \vdots & \vdots & \dots & \vdots \\ r_{n+m-1} & r_{n+m-2} & \dots & r_m \end{bmatrix} \qquad \rho_m = \begin{bmatrix} r_{n+1} \\ r_{n+2} \\ \vdots \\ r_{n+m} \end{bmatrix}. \tag{7.3.7}$$

Once the autoregressive parameters have been estimated, it is possible to compute the sequence of equation errors

$$e(t) = y(t) - \alpha_n\, y(t-1) - \dots - \alpha_1\, y(t-n) \tag{7.3.8}$$

and model this sequence by means of a MA model.

Remark 7.1.1 – It can be observed that Yule–Walker estimates belong to the family of IV methods; in fact, they use as instruments delayed outputs. The entity of the delay must be sufficient to obtain a null correlation between the instruments and equation errors but must be kept as small as possible to maximize the correlation of the instruments with current outputs in order to limit the covariance matrix of the estimate.