ST

System Theory

ST.4 LINKS BETWEEN STATE SPACE AND INPUT–OUTPUT MODELS

Consider the completely observable system (ST.3.1) and its canonical form (ST.3.12)–(ST.3.15). In this representation the whole system is decomposed into m interconnected subsystems whose orders are given by $\nu_1, \ldots, \nu_m$ The components of the state vector which define the state of the j–th subsystem are given, because of the structure of $\tilde{C}$ and $\tilde{A}$, by the following relations

$$\begin{aligned}
x_{(\nu_1+\ldots+\nu_{j-1}+1)}(t) &= y_j(t) - d_j u(t) \\
x_{(\nu_1+\ldots+\nu_{j-1}+2)}(t) &= z\, y_j(t) - b_{j1} u(t) - d_j z\, u(t) \\
x_{(\nu_1+\ldots+\nu_{j-1}+3)}(t) &= z^2 y_j(t) - b_{j2} u(t) - b_{j1} z u(t) - d_j z^2 u(t) \\
&\vdots \\
x_{(\nu_1+\ldots+\nu_j)}(t) &= z^{\nu_j-1} y_j(t) - b_{j(\nu_j-1)} u(t) - \ldots - b_{j1} z^{\nu_j-2} u(t) - d_j z^{\nu_j-1} u(t)
\end{aligned} \tag{ST.4.1}$$

where z denotes the unitary advance operator (i.e. $z\, y(t) = y(t+1)$). Relations (ST.4.1) written for $j = 1, \ldots, m$, give the whole state of the system that can be also expressed in the more compact form

$$x(t) = V(z)\, y(t) - W Z(z)\, u(t) \tag{ST.4.2}$$

where

$$V(z) = \begin{bmatrix}
1 & \ldots & 0 \\
z & & 0 \\
\vdots & & \vdots \\
z^{(\nu_1-1)} & \ldots & 0 \\
\vdots & & \vdots \\
0 & \ldots & 1 \\
0 & & z \\
\vdots & & \vdots \\
0 & \ldots & z^{(\nu_m-1)}
\end{bmatrix} \tag{ST.4.3}$$

$$Z(z) = \begin{bmatrix} I \\ z\,I \\ \vdots \\ z^{(\nu_M - 1)}\,I \end{bmatrix} \tag{ST.4.4}$$

$$\nu_M = \max_i\,(\nu_i) \tag{ST.4.5}$$

$$W = \begin{bmatrix} d_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ b_{11} & d_1 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & & \vdots \\ b_{1(\nu_1-1)} & \dots & b_{11} & d_1 & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ d_m & 0 & \dots & \dots & \dots & \dots & 0 \\ b_{m1} & d_m & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & & \vdots \\ b_{m(\nu_m-1)} & \dots & b_{m1} & d_m & 0 & \dots & 0 \end{bmatrix}_{(n \times r\nu_M)} . \tag{ST.4.6}$$

By substituting expression (ST.4.2) in equation (ST.3.1)

$$\big(z\,I - \tilde{A}\big)\,x(t) = \tilde{B}\,u(t) \tag{ST.4.7}$$

we obtain the equation

$$\big[\,(z\,I - \tilde{A})\,V(z)\,\big]y(t) = \big[\,(z\,I - \tilde{A})\,W Z(z) + \tilde{B}\,\big]\,u(t) \tag{ST.4.8}$$

constituted by n input–output relations. Only the relations in positions ν_1, $\nu_1 + \nu_2, \dots,$ $\nu_1 + \dots + \nu_m$ are, however, significant since all remaining ones are simple identities. The m significant relations can be written in the form

$$Q(z)\,y(t) = P(z)\,u(t) \tag{ST.4.9}$$

where

$$Q(z) = \big[\,q_{ij}(z)\,\big] \qquad (i,\,j = 1, \dots, m) \tag{ST.4.10}$$

$$P(z) = \big[\,p_{ij}(z)\,\big] \qquad (i = 1, \dots, m;\ j = 1, \dots, r) \tag{ST.4.11}$$

$$q_{ii}(z) = z^{\nu_i} - \alpha_{ii\nu_i}\,z^{\nu_i - 1} - \dots - \alpha_{ii2}\,z - \alpha_{ii1} \tag{ST.4.12a}$$

$$q_{ij}(z) = -\alpha_{ij\nu_{ij}}\,z^{\nu_{ij}-1} - \dots - \alpha_{ij2}\,z - \alpha_{ij1} \tag{ST.4.12b}$$

$$p_{ij}(z) = \beta_{ij(\nu_i+1)}\,z^{\nu_i} + \dots + \beta_{ij2}\,z + \beta_{ij1} \tag{ST.4.12c}$$

The coefficients of the polynomials in $Q(z)$ are those appearing in $\tilde{A}$ while those of the polynomials in $P(z)$ are linked to the entries of $\tilde{B}$ and $\tilde{D}$ by the bijection

$$\overline{B} = M\,\Phi \tag{ST.4.13}$$

$$\overline{B} = \begin{bmatrix} \overline{B}_1 \\ \vdots \\ \overline{B}_m \end{bmatrix} \qquad \overline{B}_i = \begin{bmatrix} \beta_{i11} & \dots & \beta_{ir1} \\ \vdots & & \vdots \\ \beta_{i1(\nu_i+1)} & \dots & \beta_{ir(\nu_i+1)} \end{bmatrix} \tag{ST.4.14}$$

$$\Phi = \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_m \end{bmatrix} \qquad \Phi_i = \begin{bmatrix} d_i \\ \tilde{B}_i \end{bmatrix} = \begin{bmatrix} d_{i1} & \dots & d_{ir} \\ b_{i11} & \dots & b_{ir1} \\ \vdots & & \vdots \\ b_{i1\nu_i} & \dots & b_{ir\nu_i} \end{bmatrix} \tag{ST.4.15}$$

$$M = [\, M_{ij} \,] \quad (i, j = 1, \dots, m) \tag{ST.4.16a}$$

$$M_{ii} = \begin{bmatrix} -\alpha_{ii1} & -\alpha_{ii2} & \dots & -\alpha_{ii\nu_i} & 1 \\ -\alpha_{ii2} & -\alpha_{ii3} & \dots & 1 & 0 \\ \vdots & \vdots & & & \vdots \\ -\alpha_{ii\nu_i} & 1 & & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix} (\nu_i + 1 \times \nu_i + 1) \tag{ST.4.16b}$$

$$M_{ij} = \begin{bmatrix} -\alpha_{ij1} & -\alpha_{ij2} & \dots & -\alpha_{ii\nu_{ij}} & 0 \\ -\alpha_{ij2} & -\alpha_{ij3} & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ -\alpha_{ij\nu_{ij}} & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} (\nu_i + 1 \times \nu_j + 1). \tag{ST.4.16c}$$

Remark ST.4.1 – The bijection defined by M between the entries of $P(z)$ and those of $\tilde{B}$ and $\tilde{D}$ is always well conditioned since $|\det M| = 1$, independently from the actual values of the α_{ijk} scalars. It can also be observed that the simple structure of M allows the development of *ad hoc* algorithms for its inversion.

Remark ST.4.2 – The polynomial input–output model (ST.4.9) consists in the m difference equations

$$y_i(t + \nu_i) = \sum_{j=1}^{m} \sum_{k=1}^{\nu_{ij}} \alpha_{ijk}\, y_j(t + k - 1) + \sum_{j=1}^{r} \sum_{k=1}^{\nu_i+1} \beta_{ijk}\, u_j(t + k - 1). \tag{ST.4.17}$$

Remark ST.4.3 – Relations (ST.4.9)–(ST.4.16) allow, because of the invertibility of M, two–way transformations between canonical state space models $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and MFD models $(Q(z), P(z))$. These models are characterized by the same minimal number of parameters.

Remark ST.4.4 – It can be shown that input–output models (ST.4.9) constitute a set of canonical forms with respect to the equivalence relation defined by the premultiplication of $Q(z)$ and $P(z)$ by a nonsingular unimodular matrix $M(z)$. Relations (ST.3.7) and equation (ST.4.13) lead directly to the following relations between the degrees of the entries of $Q(z)$ and $P(z)$

$$\deg q_{ii}(z) \geq \deg q_{ij}(z) \qquad \text{for } i > j \tag{ST.4.18a}$$

$$\deg q_{ii}(z) > \deg q_{ij}(z) \qquad \text{for } i < j \tag{ST.4.18b}$$

$$\deg q_{ii}(z) > \deg q_{ji}(z) \tag{ST.4.18c}$$

$$\deg q_{ii}(z) = \deg p_{ij}(z). \tag{ST.4.18d}$$

Remark ST.4.5 – Links (ST.4.13)–(ST.4.16) can be used also for purely dynamic systems, i.e. for systems where no algebraic link between input and output is present ($D = 0$). In this case the first row of the submatrices Φ_i is null and relations (ST.4.13)–(ST.4.16) can be rewritten in the simpler form that follows

$$\overline{B} = M\,\tilde{B} \tag{ST.4.19}$$

$$\overline{B} = \begin{bmatrix} \overline{B}_1 \\ \vdots \\ \overline{B}_m \end{bmatrix} \qquad \overline{B}_i = \begin{bmatrix} \beta_{i11} & \dots & \beta_{ir1} \\ \vdots & & \vdots \\ \beta_{i1\nu_i} & \dots & \beta_{ir\nu_i} \end{bmatrix} \tag{ST.4.20}$$

$$M = \left[\, M_{ij} \,\right] \quad (i, j = 1, \dots, m) \tag{ST.4.21a}$$

$$M_{ii} = \begin{bmatrix} -\alpha_{ii2} & -\alpha_{ii3} & \dots & -\alpha_{ii\nu_i} & 1 \\ -\alpha_{ii3} & -\alpha_{ii4} & \dots & 1 & 0 \\ \vdots & \vdots & & & \vdots \\ -\alpha_{ii\nu_i} & 1 & & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix} \quad (\nu_i \times \nu_i) \tag{ST.4.21b}$$

$$M_{ij} = \begin{bmatrix} -\alpha_{ij2} & -\alpha_{ij3} & \dots & -\alpha_{ii\nu_{ij}} & 0 \\ -\alpha_{ij3} & -\alpha_{ij4} & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ -\alpha_{ij\nu_{ij}} & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}_{(\nu_i \times \nu_j)} . \tag{ST.4.21c}$$

Relation (ST.4.18d) is substituted, in this case, by

$$\deg q_{ii}(z) > \deg p_{ij}(z). \tag{ST.4.22}$$

Remark ST.4.6 – The input–output canonical model $(Q(z), P(z))$ (ST.4.9) is strictly equivalent to the canonical quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. The link between the initial state of the state space model and the initial conditions of the input–output model is established by relation (ST.4.2).

Example ST.4.1 – In this example we will construct the input–output polynomial model strictly equivalent to the canonical state space model (ST.3.21). From (ST.3.21a) it follows, by inspection, that

$$\begin{aligned} \alpha_{111} &= 0.0794 & \qquad \alpha_{121} &= -0.0079 \\ \alpha_{112} &= -0.5635 & \alpha_{122} &= 0.0238 \\ \alpha_{113} &= 1.3095 & & \end{aligned}$$

$$\begin{aligned} \alpha_{211} &= 0.0238 & \qquad \alpha_{221} &= -0.1190 \\ \alpha_{212} &= -0.1190 & \alpha_{222} &= 0.6905 \\ \alpha_{213} &= 0.1429 & & \end{aligned}$$

$$Q(z) = \begin{bmatrix} z^3 - 1.3095\,z^2 + 0.5635\,z - 0.0794 & -0.0238\,z + 0.0079 \\ -0.1429\,z^2 + 0.1190\,z - 0.0238 & z^2 - 0.6905\,z + 0.1190 \end{bmatrix}.$$

$\tilde{B}$ and $\tilde{D}$ allow then to construct Φ (ST.4.15) given by

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 0 \\ 1.6667 & 0 \\ 2 & -1 \\ 3 & 1 \\ 2.1667 & 0.3333 \end{bmatrix}$$

while M (ST.4.16) is

$$M = \begin{bmatrix} -0.0794 & 0.5635 & -1.3095 & 1 & 0.0079 & -0.0238 & 0 \\ 0.5635 & -1.3095 & 1 & 0 & -0.0238 & 0 & 0 \\ -1.3095 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0238 & 0.1190 & -0.1429 & 0 & 0.1190 & -0.6905 & 1 \\ 0.1190 & -0.1429 & 0 & 0 & -0.6905 & 1 & 0 \\ -0.1429 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

$\overline{B} = M\Phi$ (ST.4.13) is

$$\overline{B} = \begin{bmatrix} -1.0873 & -0.0317 \\ 2.5159 & 0.0238 \\ -1.3095 & 0 \\ 1 & 0 \\ 0.0238 & -0.4762 \\ 1.7381 & 1.6905 \\ 1.8571 & -1 \end{bmatrix}$$

so that

$$\begin{aligned} \beta_{111} &= -1.0873 & \beta_{121} &= -0.0317 \\ \beta_{112} &= 2.5159 & \beta_{122} &= 0.0238 \\ \beta_{113} &= -1.3095 & \beta_{123} &= 0 \\ \beta_{114} &= 1 & \beta_{124} &= 0 \end{aligned}$$

$$\begin{aligned} \beta_{211} &= 0.0238 & \beta_{221} &= -0.4762 \\ \beta_{212} &= 1.7381 & \beta_{222} &= 1.6905 \\ \beta_{213} &= 1.8571 & \beta_{223} &= -1 \end{aligned}$$

and $P(z)$ is given by

$$P(z) = \begin{bmatrix} z^3 - 1.3095\,z^2 + 2.5159\,z - 0.0794 & 0.0238\,z - 0.0317 \\ 1.8571\,z^2 + 1.7381\,z + 0.0238 & -z^2 + 1.6905\,z - 0.4762 \end{bmatrix}.$$