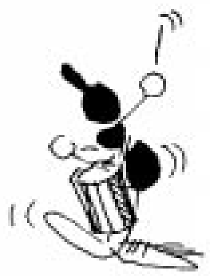
SP

Stochastic Processes



SP.3 CONVERGENCE, CONSISTENCY AND HYPOTHESIS TESTING

Definition SP.3.1 – The sequence of random variables x_N is said to converge *with probability one* or *almost surely* to x if and only if for every $\epsilon > 0$

$$\lim_{k\to\infty} \Pr\big(\,|\, x_N - x \,| \leq \epsilon,\ \forall\, N \geq k\,\big) = 1. \qquad \text{(SP.3.1)}$$

Definition SP.3.2 (Convergence in probability) – The sequence of random variables x_N is said to converge *in probability* to x if and only if for every $\epsilon > 0$

$$\Pr\big(\,|\, x_N - x \,| > \epsilon\,\big) = 0. \qquad \text{(SP.3.2)}$$

Convergence with probability one (almost sure) implies the covergence in probability but not *vice–versa*.

Definition SP.3.3 (Efficient estimates) – An estimate θ° is said to be an *efficient* estimate of a parameter θ^* if

$$\mathrm{E}\left[(\theta^\circ - \theta^*)^2\right] \leq \mathrm{E}\left[(\theta' - \theta^*)^2\right] \qquad \text{(SP.3.3)}$$

for any other estimate θ'.

Definition SP.3.4 (Consistent estimates) – Denote with θ_N° the estimate, based on N samples, of a parameter θ. θ_N° is said to be a *consistent* estimate of θ^* if

$$\lim_{N\to\infty} \mathrm{E}\left[(\theta_N^\circ - \theta^*)^2\right] = 0. \qquad \text{(SP.3.4)}$$

Consistency (in probability) of an estimate θ_N° is defined as the convergence in probability of θ_N° to θ^*, i.e.

$$\lim_{N\to\infty} \Pr\big(\,|\, \theta_N^\circ - \theta^* | > \epsilon\,\big) = 0 \quad \text{for any } \epsilon > 0. \qquad \text{(SP.3.5)}$$

A notation frequently used to denote consistency is

$$\text{plim}_{N\to\infty}\theta_N^\circ = \theta^* \tag{SP.3.6}$$

where "plim" indicates the *probability limit*. It can be shown that for any continuos function $f(\theta^*)$

$$\text{plim}\; f(\theta_N^\circ) = f(\text{plim}\,\theta_N^\circ). \tag{SP.3.7}$$

Definition SP.3.5 (Unbiased and asymptotically umbiased estimates) – An estimate θ_N° of θ^* based on N samples is said to be *unbiased* if $\mathrm{E}\,[\theta_N^\circ] = \theta^*$. The estimate is defined as *asymptotically unbiased* if

$$\lim_{N\to\infty} \mathrm{E}\left[\,\theta_N^\circ\,\right] = \theta^*. \tag{SP.3.8}$$

Definition SP.3.6 (Normal Probability Distribution) – The *normal* or *Gaussian distribution* of a random vector $x = [x_1, \ldots, x_n]^T$ with mean $\bar{x}$ and covariance matrix Σ_x, has the following structure

$$\mathrm{p}(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma_x)}} \exp\left[-\frac{(x-\bar{x})^T\;\Sigma_x^{-1}\;(x-\bar{x})}{2}\right]. \tag{SP.3.9}$$

Remark SP.3.1 – The *normal* or *Gaussian distribution* of a random variable x with mean $\bar{x}$ and variance σ_x^2 is

$$\mathrm{p}(x) = \frac{1}{\sqrt{2\pi\,\sigma_x^2}}\; \mathrm{e}^{-(x-\bar{x})^2/2\sigma_x^2}. \tag{SP.3.10}$$

Definition SP.3.7 (Chi–square distribuited random variables) – Consider an n–dimensional Gaussian random vector $u = [u_1, \ldots, u_n]^T$ characterized by

$$\mathrm{E}\left[\,u\,\right] = 0, \quad \mathrm{E}\left[\,u\,u^T\,\right] = I; \tag{SP.3.11}$$

the scalar random variable q given by the quadratic form

$$q = u^T u \tag{SP.3.12}$$

is the sum of the squares of n independent zero–mean, unitary–variance Gaussian random variables, and its probability distribution is called *chi–square* with n degrees of freedom and denoted as $\chi^2(n)$.

Remark SP.3.2 – The mean and the variance of q are given by

$$\mathrm{E}\left[\,q\,\right] = n \tag{SP.3.13}$$

$$\sigma_q^2 = 2n. \tag{SP.3.14}$$

Remark SP.3.3 – If x is an n–dimensional random Gaussian vector with mean $\bar{x}$ and covariance matrix Σ_x, then the scalar random variable

$$q = \left(x - \bar{x}\right)^T \Sigma_x^{-1} \left(x - \bar{x}\right) \tag{SP.3.15}$$

has a $\chi^2(n)$ distribution.

The probability density function previously defined are useful in the problem of hypothesis testing.

Hypothesis Testing – Hypothesis testing consists in comparing alternative statements called the *null*, H_0, and the alternative, H_1, hypotheses. For example

$$H_0 : \theta = \theta^*$$
$$H_1 : \theta \neq \theta^*$$

where θ is a certain parameter whose value is equal (not equal) to a certain value θ^* under H_0 (H_1).

Definition SP.3.8 (Type I and type II errors) – A type I error e_I is defined as

$$e_I = \Pr\left(H_1 \mid H_0 \right) = \Pr\left(\text{accept } H_1 \mid H_0 \text{ true}\right) \tag{SP.3.16}$$

while a type II error e_{II} is defined as

$$e_{II} = \Pr\left(H_0 \mid H_1 \right) = \Pr\left(\text{accept } H_0 \mid H_1 \text{ true}\right). \tag{SP.3.17}$$

Definition SP.3.9 (Power between hypotheses H_0 and H_1) – The power between hypotheses H_0 and H_1 is

$$\pi = \Pr\left(H_1 \mid H_1 \right) = 1 - e_{II} \tag{SP.3.18}$$

which is the test's capability of discerning H_1 when it is true.

A useful tool for minimizing the probability of type II error (i.e. maximizing the power of the test) can be derived from the following definition.

Definition SP.3.10 (α–percentile of a distribution) – If x is a random variable with distribution function $\mathrm{P}(x)$, the solution x_α of the equation

$$\Pr\left(x \leq x_\alpha\right) = \alpha, \qquad \alpha \in]0, 1[\tag{SP.3.19}$$

is called the α–percentile of the distribution $\mathrm{P}(x)$

The interval $[0\ x_\alpha]$ associated with a probability level α is called the $100\alpha\%$ *confidence interval* and the lower and upper bounds are the *confidence limits*.

The α–percentiles $\chi^2_\alpha(n)$ of a $\chi^2(n)$–distribuited random variable satisfy the following relation

$$\Pr\left(\chi^2(n) < \chi^2_\alpha(n)\right) = \alpha \qquad \text{(SP.3.20)}$$

and are reported in the following table.

n	$\chi^2_{.005}$	$\chi^2_{.01}$	$\chi^2_{.025}$	$\chi^2_{.05}$	$\chi^2_{.95}$	$\chi^2_{.975}$	$\chi^2_{.99}$	$\chi^2_{.995}$
1	0.00	0.00	0.001	0.004	3.84	5.02	6.63	7.88
2	0.01	0.020	0.051	0.103	5.99	7.83	9.21	10.6
3	0.072	0.115	0.216	0.352	7.81	9.35	11.3	12.8
4	0.207	0.297	0.484	0.711	9.49	11.1	13.3	14.9
5	0.412	0.554	0.831	1.15	11.1	12.8	15.1	16.7
6	0.676	0.872	1.24	1.64	12.6	14.4	16.8	18.5
7	0.989	1.24	1.69	2.17	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	18.3	20.5	23.2	25.2
20	7.43	8.26	9.58	10.9	31.4	34.2	37.6	40.0
30	13.8	15.0	16.8	18.5	43.8	47.0	50.9	53.7
40	20.7	22.1	24.4	26.5	55.8	59.3	63.7	66.8
50	28.0	29.7	32.3	34.8	67.5	71.4	76.2	79.5

Table SP.3.1 α–percentiles $\chi^2_\alpha(n)$ of the χ^2–distribution

Example SP.3.1 (Whiteness test on residuals) – The whiteness test of residuals (prediction errors) $\epsilon(t)$ is evaluated by computing the nonnegative quantity $\zeta_{N,M}$ (3.14.24) . Under the hypothesis H_0 that $\epsilon(t)$ is a white process, $\zeta_{N,M}$ has, asymptotically, a $\chi^2(M)$ distribution so that H_0 is accepted if

$$\zeta_{N,M} < \chi^2_\alpha(M)$$

where the $100\alpha\%$ confidence interval $[0\ \chi^2_\alpha(M)]$ is determined in order that

$$\Pr\left(\zeta_{N,M} < \chi^2_\alpha(M) \mid H_0\right) = \alpha. \qquad \text{(SP.3.21)}$$

For example the 99% confidence interval (i.e. $\alpha = 0.99$) when $M = 8$, is [0 20.1] (see table (SP.3.1)). $1 - \alpha$ is called *significance level* of the test.