Lecture 2.5

- Nonparametric Methods
- Input Signals
- Model Parameterizations

System Identification

Obtain a model of a system from measured inputs and outputs.

Type of model depends on application and system. Often we assume that the true system can be described as a LTI (linear time-invariant) system:

$$y(t) = G_0(q)u(t) + v(t) \tag{1a}$$

or, equivalently,

$$y(t) = \sum_{k=1}^{\infty} g_0(k)u(t-k) + v(t) \tag{1b}$$

Question: How do we determine the model $G_0(q)$ or $\{g_0(k)\}$?

Parametric models:

Postulate a model $G(q, \boldsymbol{\theta})$ parameterized by $\boldsymbol{\theta}$.

- Easy to use for simulation, control design, etc.
- Often accurate models.
- Requires some work...
- Example: FIR model

$$y(t) = u(t) + b_1 u(t-1) + b_2 u(t-2)$$
$$\Rightarrow G(q^{-1}, \boldsymbol{\theta}) = 1 + b_1 q^{-1} + b_2 q^{-2}, \quad \boldsymbol{\theta} = [b_1 \, b_2]^T$$

Question: Can we determine $G_0(q)$ or $\{g_0(k)\}$ without postulating a parameterized model?

Nonparametric Identification

Nonparametric models:

Determine G_0 or $\{g_0(k)\}$ without parameterizing.

- Simple to obtain.
- Results often in graphs or tables which can not easily be used for simulation, etc.
- Often used to validate parametric models.
- Transient analysis, correlation analysis, frequency analysis, spectral analysis.

Transient Analysis

Impulse response analysis: Applying the input

$$u(t) = \begin{cases} k, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

to (1b) gives the output

$$y(t) = k g_0(t) + v(t)$$

which motivates the impulse response estimate

$$\hat{g}(t) = \frac{y(t)}{k}$$

Step-response analysis Applying the input

$$u(t) = \begin{cases} k, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

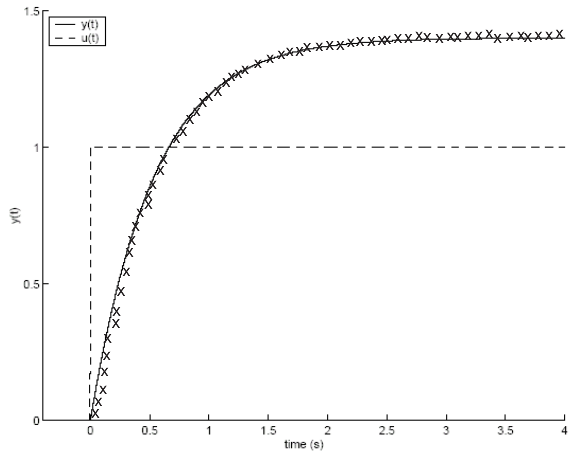
gives the output

$$y(t) = k \sum_{k=1}^{t} g_0(k) + v(t)$$

which motivates the impulse response estimate

$$\hat{g}(t) = \frac{y(t) - y(t-1)}{k}$$

Ex: Step-response (true – solid, measured – $\times$)



Transient analysis

- Input taken as impulse or step.
- Model consists of recorded output, or an estimate of $g_0(k)$.
- Convenient for deriving crude models. Gives estimates of dominating time constants, time delays and static gain.
- Sensitive to noise.
- Poor excitation.

Correlation Analysis

System:

$$y(t) = \sum_{k=1}^{\infty} g_0(k)u(t-k) + v(t)$$

where $u(t)$ is a stochastic process which is independent of $v(t)$. Multiplying by $u(t-\tau)$ and taking expectation yields

$$r_{yu}(\tau) = \sum_{k=1}^{\infty} g_0(k) r_u(\tau - k)$$

which is known as the Wiener-Hopf equation.

In practice, truncate the sum and solve the resulting system of eq.

$$\hat{r}_{yu}(\tau) = \sum_{k=1}^{M} \hat{g}(k)\hat{r}_u(\tau - k)$$

Estimates of the covariance functions.

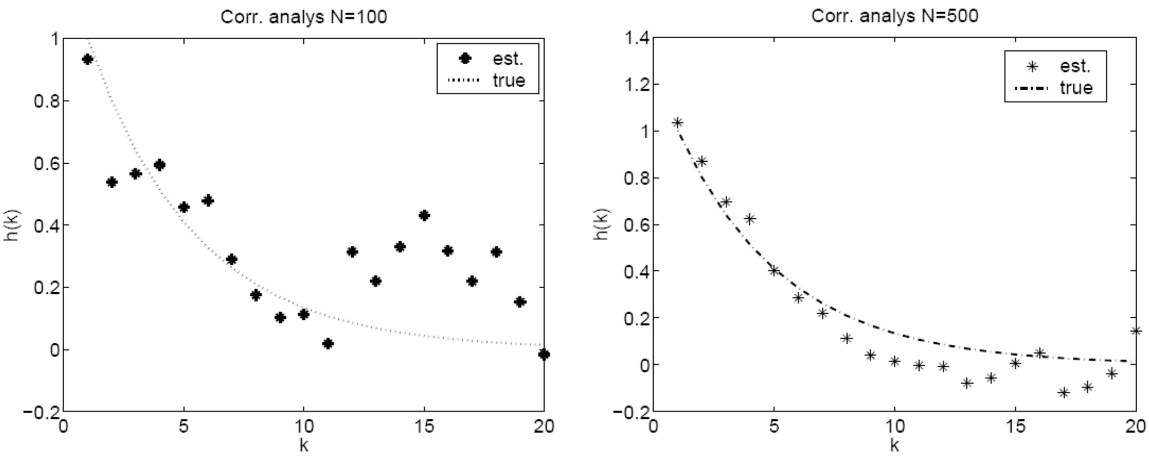
- First choice:

$$\hat{r}_{yu}(\tau) = \frac{1}{N} \sum_{k=1}^{N-\tau} y(k+\tau)u(k) \quad (\tau \geq 0)$$

- Second choice:

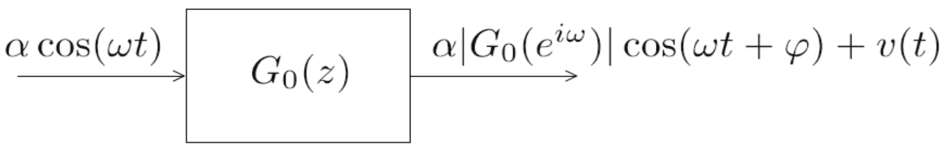
$$\hat{r}_{yu}(\tau) = \frac{1}{N-\tau} \sum_{k=1}^{N-\tau} y(k+\tau)u(k) \quad (\tau \geq 0)$$

Which one to prefer?



Frequency Analysis

Estimate $G_0(e^{i\omega})$!



- Repeat experiment for different ω $(t = 1, \ldots, N)$.
- Determine the phase shift and the amplitude of the output.
- Results in a Bode plot ($|G_0(e^{i\omega})|$ and $\arg G_0(e^{i\omega})$).
- Sensitive to noise. Require long experiments.
- Gives basic information about the system.

Spectral Analysis

- The correspondence of the Wiener-Hopf equation in the frequency domain is given by:

$$\Phi_{yu}(\omega) = G(e^{-i\omega})\Phi_u(\omega)$$

- An estimate of the transfer function can be obtained as:

$$\hat{G}(e^{-i\omega}) = \hat{\Phi}_{yu}(\omega)/\hat{\Phi}_u(\omega)$$

- Use estimates of the spectral densities, *e.g.*,

$$\hat{\Phi}_{yu}(\omega) = \frac{1}{2\pi N}\sum_{\tau=-N}^{N} \hat{r}_{yu}(\tau)e^{-i\tau\omega}$$

- Errors in $\hat{r}_{yu}(\tau)$ are summed together $\Rightarrow$ not consistent!
  - N large $\Rightarrow$ total (square) error is large even if the error in $\hat{r}_{yu}(\tau)$ is small for all τ.
  - $\hat{r}_{yu}(\tau)$ decreases slowly $\Rightarrow$ poor estimate of $\hat{r}_{yu}(\tau)$ for large values of τ.
- Better estimates are obtained if a lag window, $w(t)$, is used:

$$\hat{\Phi}_{yu}(\omega) = \frac{1}{2\pi}\sum_{\tau=-N}^{N} \hat{r}_{yu}(\tau)w(\tau)e^{-i\tau\omega}$$

- Length of lag window (M) - compromise between bias and variance (high resolution and reducing erratic fluctuations).

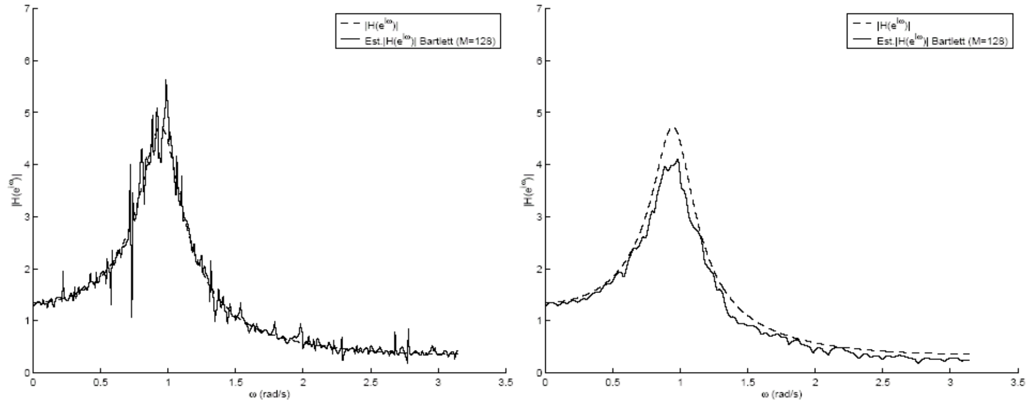


Figure 1: Spectral analysis, $N = 256$: Left: Periodogram. Right: Bartlett window $M = 128$.

Summary - Nonparametric Methods

- Results often in graph or table (step response, weighting function, transfer function etc.).
- Transient analysis (step-response, impulse response).
- Frequency analysis (sinusoidal input).
- Correlation analysis (weighting function estimate).
- Spectral analysis (transfer function estimate).
- Useful for obtaining crude estimates of time constants, cut-off frequencies etc. or for model validation.

Input Signals

The quality of the model is dependent on an appropriate choice of input signal.

Examples of useful input signals are:

- Step function.
- Pseudorandom binary sequence (PRBS).
- Sums of sinusoids

Most often the input signal is characterized by its first and second order moments:

$$m = Eu(t)$$
$$r(\tau) = E(u(t+\tau) - m)(u(t) - m)^T$$

and/or its spectral density:

$$\Phi(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i\tau\omega}$$

Rem: Deterministic signals

$$Eu(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)$$

Step Function

$$u(t) = \begin{cases} k, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Properties

- Mostly used for transient analysis: overshoot, static gain, major time constants.
- Limited usability for parametric modeling.

PseudoRandom Binary Sequence (PRBS)

A PRBS $u(t)$ is a periodic, deterministic signal with white-noise-like properties.

$$u(t) = \text{rem}(A(q)u(t), 2) = \text{rem}(a_1 u(t-1) + \cdots + a_n u(t-n), 2)$$

Properties

- The signal shifts between two levels in a certain fashion depending on $A(q)$.
- Spectral characteristics is determined by $A(q)$ and, in particular, by the period length $M = 2^n - 1$.
- Deterministic sequence behaving as noise (reproducibility).

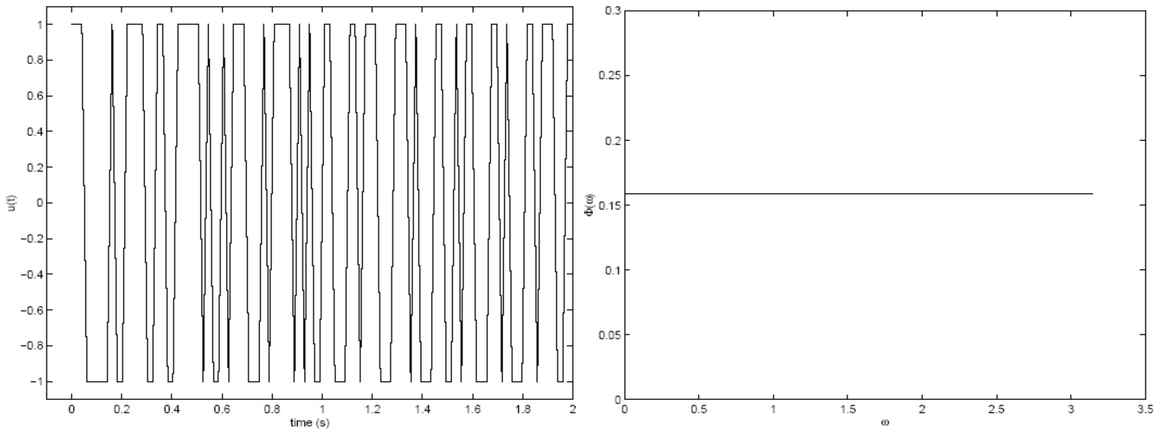


Figure 2: PRBS sequence, p=0.5, $M = \infty$. Left: Example of realization. Right: Spectral density.

Sum of Sinusoids

$$u(t) = \sum_{m=1}^{M} a_m \sin(\omega_m t + \varphi_m)$$

Properties

- User parameters: a_m, ω_m and φ_m.
- Covariance function given by:

$$r(\tau) = \sum_{m=1}^{M} \frac{a_m^2}{2} \cos(\omega_m t + \varphi_m)$$

- Spectral density given by:

$$\Phi(\omega) = \sum_{m=1}^{M} \frac{a_m^2}{4} [\delta(\omega - \omega_m) + \delta(\omega + \omega_m)]$$

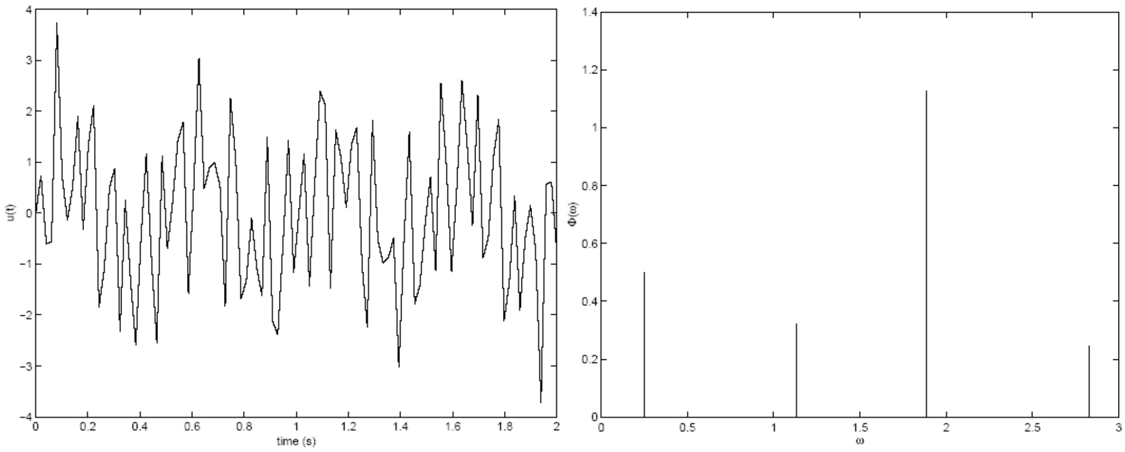


Figure 4: Sum of 4 sinusoids. Left: Signal. Right: Spectral density.

Persistent Excitation

To obtain estimates of a parametric model the input signal has to be "rich" enough to excite all modes of the system.

A input signal is said to be persistently exciting ($p.e.$) of order n if:

(i) The following limit exists:

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t+\tau)u^T(t)$$

Rem: $u(t)$ ergodic implies

$$r_u(\tau) = E\, u(t+\tau)u^T(t)$$

(ii) The matrix:

$$\mathbf{R}_u(n) = \begin{pmatrix} r_u(0) & r_u(1) & \cdots & r_u(n-1) \\ r_u(-1) & r_u(0) & & \vdots \\ \vdots & & \ddots & \\ r_u(1-n) & \cdots & & r_u(0) \end{pmatrix}$$

is positive definite.

- Another definition: $\det \mathbf{R}_u(n) \neq 0$.
- And another: $u(t)$ is *p.e.* of order n if $\Phi_u(\omega) \neq 0$ on at least n points in the interval $-\pi < \omega \leq \pi$.

An input signal that is *p.e.* of order $2n$ can be used to consistently estimate a parametric model of order $\leq n$.

- A step function is *p.e.* of order 1.
- A PRBS with period M is *p.e.* of order M.
- A sum of m sinusoids is *p.e.* of order $2m$ (if $\omega_m \neq 0$ and $\omega_m \neq \pi$).

Another important observation!

A parametric model becomes more accurate in the frequency region where the input signal has the major part of its energy.

A physical process is often of low frequency character $\Rightarrow$ use low-pass filtered signal as input.

Summary - Input Signals

- The choice of input signal determines the quality of the final parametric model.
- The obtained parametric model is more accurate in frequency regions where the input signal contains much energy.
- An input signal has to be rich enough to excite all interesting modes of the system (persistently exciting of sufficiently high order).
- In practice there might be some restrictions on the input.