Recursive (On-line) Identification Methods

- Recursive Least Squares (RLS) Methods
- Forgetting Factor and Tracking Time-Varying Parameters
- Identification condition problem
- Computational Aspects

Course Outline

- Introduction and overview on system identification
- Non-recursive (off-line) identification methods
- *Recursive (on-line) identification methods (I)*
- Recursive (on-line) identification methods (II)
- Practical aspects and applications of system identification

Why?

Why is recursive identification of interest?

- On-line Estimation.
- Adaptive Systems.
- Time Varying Parameters.
- Fault Detection and Diagnosis.
- Simulation.

How?

How do we estimate time-varying parameters?

- Update the model regularly (once every sampling instant)
- Make use of previous calculations in an efficient manner.
- The basic procedure is to modify the corresponding off-line method, *e.g.*, the block/batch least squares method, the prediction error method.

Desirable Properties

We desire our recursive algorithms to have the following properties:

- Fast convergence.
- Consistent estimates (time-invariant models).
- Good tracking (for time-varying parameters, e.g. in the event of fault occurrence or operating condition changes).
- Computationally simple (perform all calculations during one sampling interval).

Trade-offs

No algorithm is perfect. The design is always based on trade-offs, such as:

- Convergence versus tracking.
- Computational complexity versus accuracy.

Recursive Least Squares Method (RLS)

$$\hat{\boldsymbol{\theta}}(t) = \arg\min_{\boldsymbol{\theta}} V_t(\boldsymbol{\theta}), \quad V_t(\boldsymbol{\theta}) = \sum_{k=1}^{t} \varepsilon^2(k)$$

where $\varepsilon(k) = y(k) - \boldsymbol{\varphi}^T(k)\boldsymbol{\theta}$. The solution reads:

$$\hat{\boldsymbol{\theta}}(t) = \boldsymbol{R}_t^{-1}\boldsymbol{r}_t$$

where

$$\boldsymbol{R}_t = \sum_{k=1}^{t} \boldsymbol{\varphi}(k)\boldsymbol{\varphi}^T(k), \qquad \boldsymbol{r}_t = \sum_{k=1}^{t} \boldsymbol{\varphi}(k)y(k)$$

- The criterion function $V_t(\boldsymbol{\theta})$ changes every time step, hence the estimate $\hat{\boldsymbol{\theta}}(t)$ changes every time step.
- How can we find a recursive implementation of $\hat{\boldsymbol{\theta}}(t)$?

RLS

Algorithm:

At time $t = 0$: Choose initial values of $\hat{\boldsymbol{\theta}}(0)$ and $\boldsymbol{P}(0)$

At each sampling instant, update $\boldsymbol{\varphi}(t)$ and compute

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t-1) + \boldsymbol{K}(t)\varepsilon(t)$$

$$\varepsilon(t) = y(t) - \boldsymbol{\varphi}^T(t)\hat{\boldsymbol{\theta}}(t-1)$$

$$\boldsymbol{K}(t) = \boldsymbol{P}(t)\boldsymbol{\varphi}(t)$$

$$\boldsymbol{P}(t) = \left[\boldsymbol{P}(t-1) - \frac{\boldsymbol{P}(t-1)\boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t)\boldsymbol{P}(t-1)}{1 + \boldsymbol{\varphi}^T(t)\boldsymbol{P}(t-1)\boldsymbol{\varphi}(t)}\right]$$

Question: How to obtain/derive this recursive version of LS from the block/batch LS?

Tracking

How do we handle time-varying parameters? — two ways:

- Postulate a time-varying model for the parameters. Typically we let the parameters vary according to a random walk and use the Kalman filter as an estimator.
- Modify the cost function so that we gradually forget old data. Hence, the model is fitted to the most recent data (the parameters are adapted to describe the newest data).

- Modified cost function:

$$\hat{\boldsymbol{\theta}}(t) = \arg\min_{\boldsymbol{\theta}} V_t(\boldsymbol{\theta}), \quad V_t(\boldsymbol{\theta}) = \sum_{k=1}^{t} \beta(t,k)\varepsilon^2(k)$$

- Suppose that the weighting function $\beta(t,k)$ satisfies

$$\begin{aligned} \beta(t,k) &= \lambda(t)\beta(t-1,k), \quad 0 \leq k < t \\ \beta(t,t) &= 1 \end{aligned}$$

  A common choice is to let $\lambda(t) = \lambda$, where λ is referred to as a so-called *forgetting factor*. In this case we get:

$$\beta(t,k) = \lambda^{t-k}, \quad 0 < \lambda \leq 1$$

- $\lambda = 1$ corresponds to the standard RLS.

Weighted RLS

Algorithm:

At time $t = 0$: Choose initial values of $\hat{\boldsymbol{\theta}}(0)$ and $\boldsymbol{P}(0)$

At each sampling instant, update $\boldsymbol{\varphi}(t)$ and compute

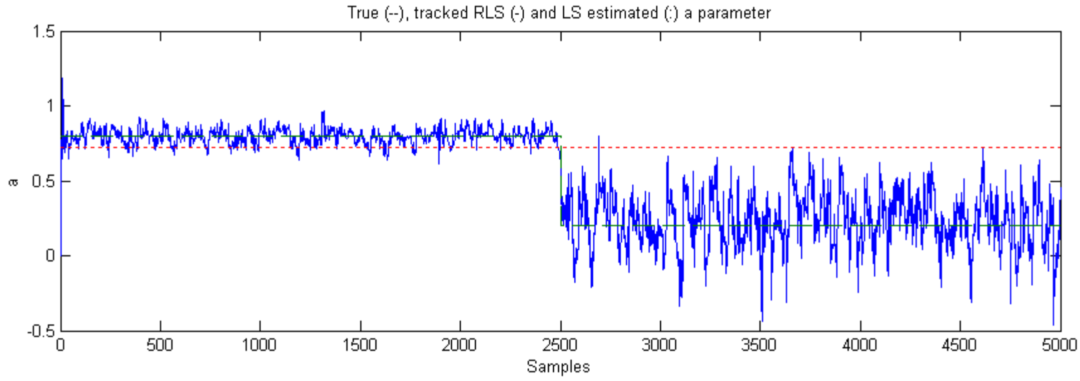
$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t-1) + \boldsymbol{K}(t)\varepsilon(t)$$

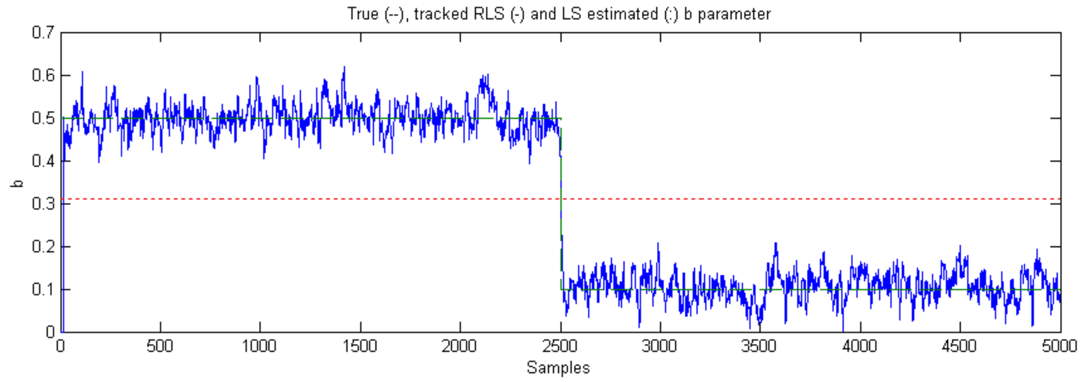
$$\varepsilon(t) = y(t) - \boldsymbol{\varphi}^T(t)\hat{\boldsymbol{\theta}}(t-1)$$

$$\boldsymbol{K}(t) = \boldsymbol{P}(t)\boldsymbol{\varphi}(t)$$

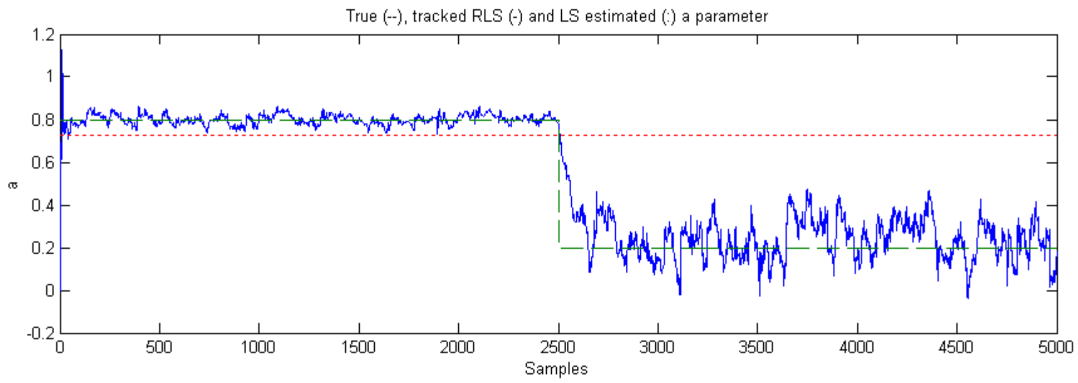
$$\boldsymbol{P}(t) = \frac{1}{\lambda(t)}\left[\boldsymbol{P}(t-1) - \frac{\boldsymbol{P}(t-1)\boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t)\boldsymbol{P}(t-1)}{\lambda(t) + \boldsymbol{\varphi}^T(t)\boldsymbol{P}(t-1)\boldsymbol{\varphi}(t)}\right]$$

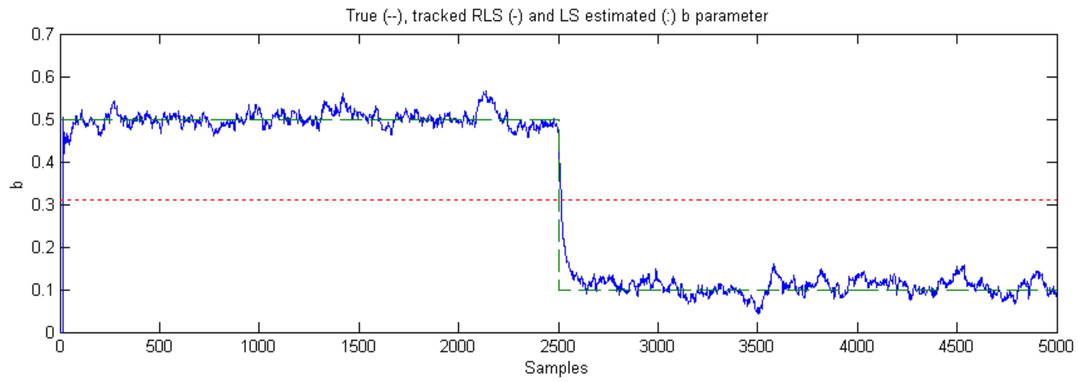
$\lambda = 0.90$



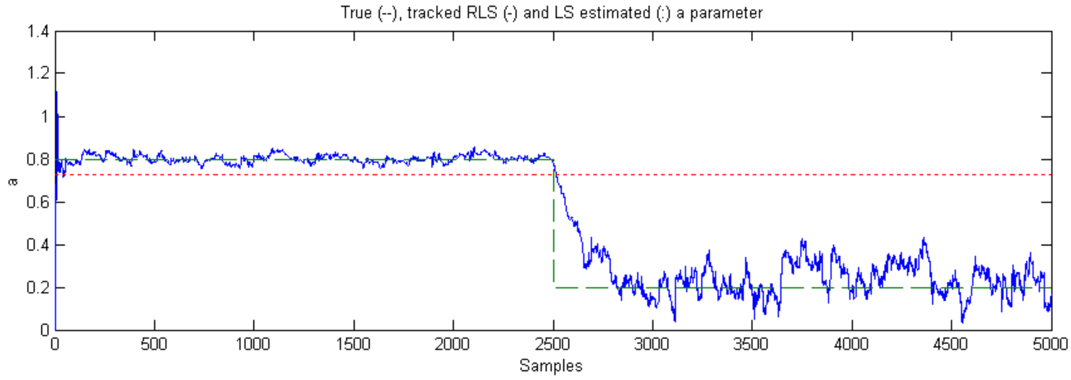


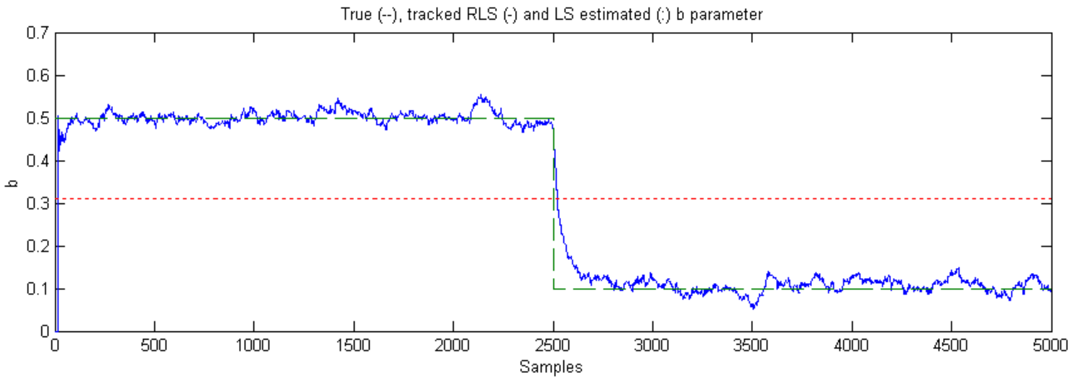
$\lambda = 0.97$



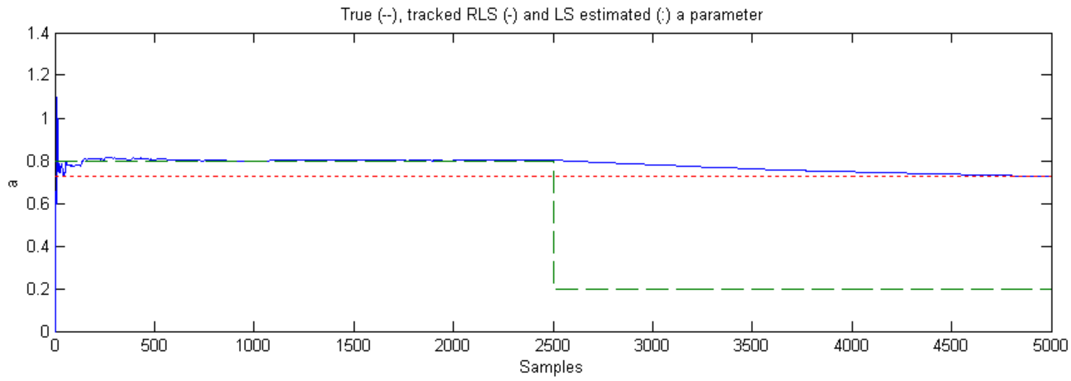


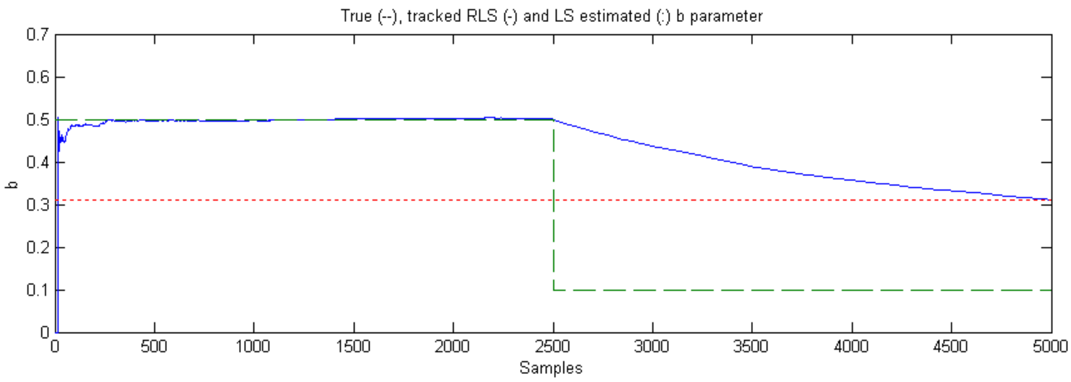
$\lambda = 0.98$





$\lambda = 1.0$





Initial Conditions

- $\hat{\boldsymbol{\theta}}(0)$ is the initial parameter estimate.
- View $\boldsymbol{P}(0)$ as an estimate of the covariance matrix of the initial parameter estimate.
  - $\boldsymbol{P}(0)$ (and $\boldsymbol{P}(t)$) are covariance matrices, and must be symmetric and positive definite.
  - Choose $P(0) = \rho \boldsymbol{I}$.
  - ρ large $\Rightarrow$ large initial response. Good if initial estimate $\hat{\boldsymbol{\theta}}(0)$ is uncertain.

Forgetting Factor

Let $\lambda(t) = \lambda$. The forgetting factor λ will then determine the tracking capability.

- We must have $\lambda = 1$ to get convergence.
- λ small $\Rightarrow$ Old data is forgotten faster, hence better tracking.
- λ small $\Rightarrow$ the algorithm is more sensitive to noise (bad convergence).
- The memory constant is defined as $T_0 = \frac{1}{1-\lambda}$. If $\lambda = 0.95, T_0 = 20$

The choice of λ is consequently a trade-off between tracking capability and noise sensitivity. A typical choice is $\lambda \in (0.95, 0.99)$. It is common to let $\lambda(t)$ tend exponentially to 1, *e.g.*,

$$\lambda(t) = 1 - \lambda_0^t(1 - \lambda(0))$$

Conclusions

- In practical scenarios, one often need to use recursive identification (time-varying systems, online identification, fault diagnosis).
- Both the LS and the IVM can easily be recast in recursive forms. The PEM can only be approximated.
- The properties of the on-line methods are comparable with the off-line case.
- Tracking capability can be incorporated by using forgetting factor techniques, or by model the parameter variations.
- There is always a tradeoff between convergence speed and tracking properties, as well as computational complexity and accuracy.
- In practice, one can make simplifications and modifications to make the recursion cheaper and more numerically robust.