The algorithm derivation below can be found in Brierley [1] and Brierley and Batty [2]. Please refer to these for a hard copy.

Back Propagation Weight Update Rule

This idea was first described by Werbos [3] and popularised by Rumelhart *et al.*[4].

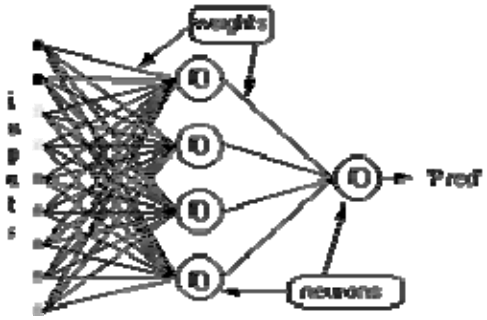


Fig 1 A multilayer perceptron

Consider the network above, with one layer of hidden neurons and one output neuron. When an input vector is propagated through the network, for the current set of weights there is an output *Pred*. The objective of supervised training is to adjust the weights so that the difference between the network output *Pred* and the required output *Req* is reduced. This requires an algorithm that reduces the absolute error, which is the same as reducing the squared error, where:

$$\text{Network Error} = Pred - Req = E \tag{1}$$

The algorithm should adjust the weights such that E^2 is minimised. Back-propagation is such an algorithm that performs a gradient descent minimisation of E^2.

In order to minimise E^2, its sensitivity to each of the weights must be calculated. In other words, we need to know what effect changing each of the weights will have on E^2. If this is known then the weights can be adjusted in the direction that reduces the absolute error.

The notation for the following description of the back-propagation rule is based on the diagram below.

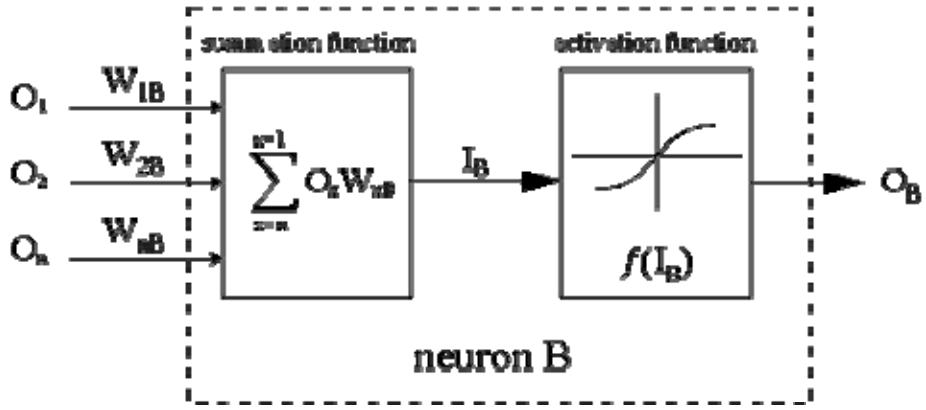


Fig 2 notation used

The dashed line represents a neuron *B*, which can be either a hidden or the output neuron. The outputs of *n* neurons ($O_1 \ldots O_n$) in the preceding layer provide the inputs to neuron *B*. If neuron *B* is in the hidden layer then this is simply the input vector.

These outputs are multiplied by the respective weights ($W_{1B} \ldots W_{nB}$), where W_{nB} is the weight connecting neuron *n* to neuron *B*. The summation function adds together all these products to provide the input, I_B, that is processed by the activation function $f(.)$ of neuron *B*. $f(I_B)$ is the output, O_B, of neuron *B*.

For the purpose of this illustration, let neuron 1 be called neuron *A* and then consider the weight W_{AB} connecting the two neurons.

The approximation used for the weight change is given by the delta rule:

$$W_{AB(new)} = W_{AB(old)} - \eta \frac{\partial E^2}{\partial W_{AB}} \quad (2)$$

Where η is the learning rate parameter, which determines the rate of learning, and

$$\frac{\partial E^2}{\partial W_{AB}}$$

is the sensitivity of the error, E^2, to the weight W_{AB} and determines the direction of search in weight space for the new weight $W_{AB(new)}$ as illustrated in the figure below.

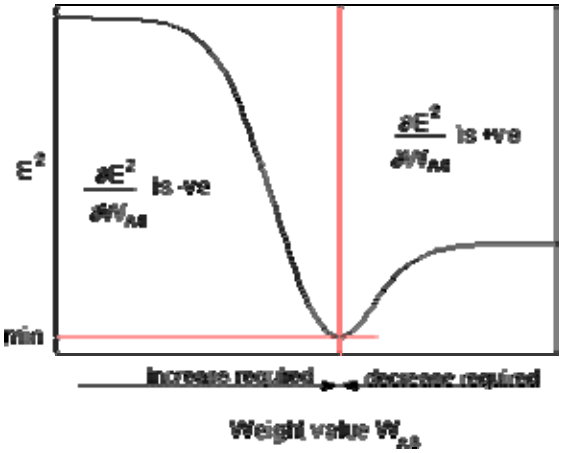


Fig 3 In order to minimise E^2 the delta rule gives the direction of weight change required

From the chain rule,

$$\frac{\partial E^2}{\partial W_{AB}} = \frac{\partial E^2}{\partial I_B}\frac{\partial I_B}{\partial W_{AB}} \tag{3}$$

and

$$\frac{\partial I_B}{\partial W_{AB}} = \frac{\partial \sum_{x=n}^{x=1} O_x W_{xB}}{\partial W_{AB}}$$

$$= \frac{\partial (O_A W_{AB})}{\partial W_{AB}} + \frac{\partial \sum_{X=n}^{X=2} O_x W_{xB}}{\partial W_{AB}}$$

$$= O_A \tag{4}$$

since the rest of the inputs to neuron *B* have no dependency on the weight W_{AB}.

Thus from eqns. (3) and (4), eqn. (2) becomes,

$$W_{AB(new)} = W_{AB(old)} - \eta \frac{\partial E^2}{\partial I_B} O_A \tag{5}$$

and the weight change of W_{AB} depends on the sensitivity of the squared error, E^2, to the input, I_B, of unit B and on the input signal O_A.

There are two possible situations:

1. B is the output neuron;
2. B is a hidden neuron.

Considering the first case:

Since B is the output neuron, the change in the squared error due to an adjustment of W_{AB} is simply the change in the squared error of the output of B:

$$\partial E^2 = \partial(\text{Pred-Req})^2$$

$$\frac{\partial E^2}{\partial I_B} = 2(\text{Pred-Req}) \frac{\partial \text{Pred}}{\partial I_B}$$

$$= 2E \frac{\partial f(I_B)}{\partial I_B}$$

$$= 2Ef'(I_B) \tag{6}$$

Combining eqn. (5) with (6) we get,

$$W_{AB(new)} = W_{AB(old)} - \eta O_A 2Ef'(I_B) \tag{7}$$

the rule for modifying the weights when neuron B is an output neuron.

If the output activation function, $f(.)$, is the logistic function then:

$$f(x) = \frac{1}{1+e^{-x}} = (1+e^{-x})^{-1} \tag{8}$$

differentiating (8) by its argument x;

(9)

But,

$$f(x) = \frac{1}{1+e^{-x}}$$

$$\Rightarrow \quad e^{-x} = \frac{(1-f(x))}{f(x)}$$

(11)

inserting (11) into (9) gives:

$$f'(x) = \frac{(1-f(x))}{f(x)} \Big/ \frac{1}{(f(x))^2}$$

$$= f(x) \times (1-f(x))$$

(12)

similarly for the tanh function,

$$f'(x) = (1-f(x)^2)$$

or for the linear (identity) function,

$$f'(x) = 1$$

This gives:

$$W_{AB(new)} = W_{AB(old)} - \eta O_A \, 2E \, O_B (1-O_B) \qquad \text{(logistic)}$$

$$W_{AB(new)} = W_{AB(old)} - \eta O_A \, 2E \, (1-O_B^2) \qquad \text{(tanh)}$$

$$W_{AB(new)} = W_{AB(old)} - \eta O_A \, 2E \qquad \text{(linear)}$$

Considering the second case:

B is a hidden neuron.

$$\frac{\partial E^2}{\partial I_B} = \frac{\partial E^2}{\partial I_0}\frac{\partial I_0}{\partial O_B}\frac{\partial O_B}{\partial I_B}$$

(13)

where the subscript, *o*, represents the output neuron.

$$\frac{\partial O_B}{\partial I_B} = \frac{\partial f(I_B)}{\partial I_B} = f'(I_B)$$

$$\frac{\partial I_0}{\partial O_B} = \frac{\partial \sum_p O_p W_{p0}}{\partial O_B}$$

(15)

where *p* is an index that ranges over all the neurons including neuron *B* that provide input signals to the output neuron. Expanding the right hand side of equation (15),

$$\frac{\partial \sum_p O_p W_{p0}}{\partial O_B} = \frac{\partial O_B W_{B0}}{\partial O_B} + \frac{\partial \sum_{p \neq B} O_p W_{p0}}{\partial O_B} = W_{B0}$$

(16)

since the weights of the other neurons ,W_{pO} (p!=B) have no dependency on O_B.

Inserting (14) and (16) into (13),

$$\frac{\partial E^2}{\partial I_B} = \frac{\partial E^2}{\partial I_0} W_{B0} f'(I_B)$$

(17)

Thus is now expressed as a function of , calculated as in (6).

The complete rule for modifying the weight W_{AB} between a neuron *A* sending a signal to a neuron *B* is,

$$W_{AB(new)} = W_{AB(old)} - \eta \frac{\partial E^2}{\partial I_B} O_A$$

(18)

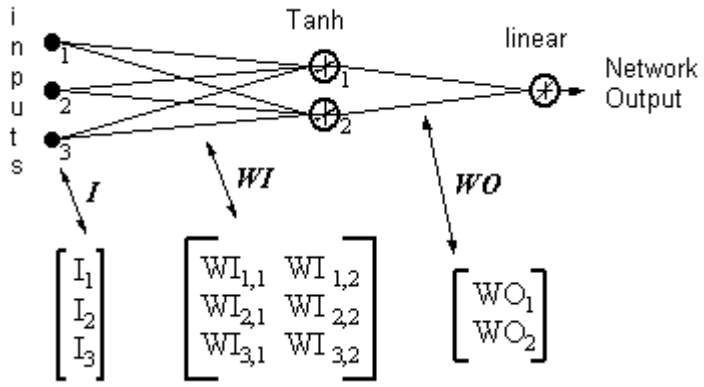
where,

$$\frac{\partial E^2}{\partial I_B} = 2E\, f_o'(I_B)$$ -I_B is the output neuron

$$\frac{\partial E^2}{\partial I_B} = \frac{\partial E^2}{\partial I_O} W_{BO}\, f_h'(I_B)$$ -I_B is a hidden neuron

where $f_o(.)$ and $f_h(.)$are the output and hidden activation functions respectively.

Example



Network Output = [tanh(I^T .WI)] . WO

let

HID = [Tanh(I^T.WI)]^T - the outputs of the hidden neurons

ERROR = (Network Output - Required Output)

LR = learning rate

The weight updates become,

linear output neuron

WO = WO - (LR x ERROR x HID)

(21)

tanh hidden neuron

WI = WI - { LR x [ERROR x WO x (1- HID^2)] . I^T }^{T}

(22)

Equations 21 and 22 show that the weights change is an input signal multiplied by a local gradient. This gives a direction that also has magnitude dependent on the magnitude of the error. If the direction is taken with no magnitude then all changes will be of equal size which will depend on the learning rate.

The algorithm above is a simplified version in that there is only one output neuron. In the original algorithm more than one output is allowed and the gradient descent minimises the total squared error of all the outputs. With only one output this reduces to minimising the error.

There are many algorithms that have evolved from the original algorithm with the aim to increase the learning speed. These are summarised in [5].

References

[1] P.Brierley, Appendix A in "Some Practical Applications of Neural Networks in the Electricity Industry" Eng.D. Thesis, 1998, Cranfield University, UK.

[2] P.Brierley and B.Batty, "Data mining with neural networks - an applied example in understanding electricity consumption patterns" in "Knowledge Discovery and Data Mining" (ed Max Bramer) 1999, chapter 12, pp.240-303, IEE, ISBN 0 85296 767 5.

[3] P.J. Werbos, "Beyond regression: New tools for prediction and analysis in the behavioural sciences," Ph.D. Thesis, 1974, Harvard University, Cambridge, MA.

[4] D.E. Rumelhart, G.E. Hinton and R.J. Williams, "Learning internal representation by error propagation," In Parallel Distributed Processing: Exploration in the Microstructure of Cognition (D.E Rumelhart and J.L. McClelland, eds.) 1986, vol. 1, chapter 8, Cambridge, MA, MIT Press.

[5] "Back Propagation family album" - Technical report C/TR96-05, Department of Computing, Macquarie University, NSW, Australia. www.comp.mq.edu.au/research.html